93. On the Darboux Transformation of Second Order Ordinary Differential Operator

By Mayumi OHMIYA
Department of Mathematics, Tokushima University

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1. Introduction. The main purpose of the present paper is to study the relation between the *Darboux transformation* of 2-nd order ordinary differential operator and the recursion formula called the *Lenard relation.* The Darboux transformation is studied in [1], [3], [4], [5] and [6] for the 1-dimensional Schrödinger operator and the ordinary differential operator of Fuchsian type. In this paper we supplement them with the description of more general aspect of the theory.

2. Darboux transformation. Consider the 2-nd order ordinary differential operator $L(u)=\partial^2-u(x)$, $\partial=d/dx$, where $u(x)$ is a complex analytic function defined in a region $\Omega\subset\boldsymbol{P}_1$. Suppose that $u(x)$ is holomorphic at $x=a\in\Omega$ and let $y_j(x)$ $(j=1,2)$ be the fundamental system of solutions of the differential equation

$$L(u)y=y''-u(x)y=0, \qquad '=d/dx$$

such that $W(y_1(a), y_2(a))=E$, where $W(f,g)=\begin{pmatrix} f & g \\ f' & g' \end{pmatrix}$ is the Wronskian matrix and E is the unit matrix of size 2. For $\zeta=[\xi_1:\xi_2]\in\boldsymbol{P}_1$, put $v(x,\zeta)=(\partial/\partial x)\log(\xi_1 y_1(x)+\xi_2 y_2(x))$ and $A_{\pm}(\zeta)=\partial\pm v(x,\zeta)$. Then the factorization $L(u)=A_+(\zeta)A_-(\zeta)$ follows. On the other hand, put

$$L^*(u;\zeta)=A_-(\zeta)A_+(\zeta).$$

$L^*(u;\zeta)$ is the 2-nd order ordinary differential operator parametrized by $\zeta\in\boldsymbol{P}_1$. We call $L^*(u;\zeta)$ the Darboux transformation of $L(u)$. Put $u^*(x,\zeta)=u(x)-2(\partial/\partial x)v(x,\zeta)$, which is analytic in $\Omega^*=\Omega\backslash\{\text{zeros of } \sum_{j=1}^2\xi_j y_j(x)\}$, then $L^*(u;\zeta)=\partial^2-u^*(x,\zeta)$ follows.

3. Lenard relation. Define the function $Q_n(x)$ $(n=1,2,\cdots)$ by the recursion formula

$$2Q'_{n+1}(x)=u'(x)Q_n(x)+2u(x)Q'_n(x)-2^{-1}Q'''_n(x)$$

with $Q_0(x)=1$. It is known that $Q_n(x)$ are polynomials of $u, u', \cdots, u^{(2n-2)}$ with constant coefficients (cf. [7]). Of course, while an arbitrary constant appears when we integrate $Q'_n(x)$ to obtain $Q_n(x)$ itself, we can define uniquely $Q_n(x)$ by putting them zero. Hence we can define the nonlinear differential operators $Z_n(u)$ and $X_n(u)$ by $Z_n(u(x))=2Q_n(x)$ and $X_n(u(x))=2Q'_n(x)=\partial Z_n(u(x))$. Then we can rewrite the above recursion formula as

$$X_n(u)=(2^{-1}u'+u\partial-4^{-1}\partial^3)Z_{n-1}(u),$$

which is called the Lenard relation. $Z_n(u)$ turns out to be the $(2n-2)$-th order differential polynomial. For example, we have