64. A Generalization of the Hille-Yosida Theorem

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1. Introduction. Let X be a Banach space, and let $B(X)$ be the set of all bounded linear operators from X into itself. Arendt [1] introduced the notion of integrated semigroups and obtained the following generalization of the Hille-Yosida theorem: A closed linear operator A is the generator of a once integrated semigroup $\{U(t); t\geq 0\}$ on X satisfying $\|U(t+h)-U(t)\|\leq Mhe^{a(t+h)}$ for $t, h\geq 0$ if and only if $(a, \infty)\subset\rho(A)$ and $\|(\lambda-A)^{-m}\|\leq M/(\lambda-a)^m$ for $\lambda>a$ and $m\geq 1$, where $M>0$ and $a\geq 0$ are constants. Moreover, the part of A in $\overline{D(A)}$ is the generator of a (C_0)-semigroup on $\overline{D(A)}$.

Let $C\in B(X)$ be injective. In this paper we introduce the notion of integrated C-semigroups and prove the following theorems.

Theorem 1. *An operator A is the generator of an integrated C-semigroup $\{U(t); t\geq 0\}$ on X satisfying*

(1.1) $$\|U(t+h)-U(t)\|\leq Mhe^{a(t+h)} \qquad \textit{for } t, h\geq 0,$$

where $M>0$ and $a\geq 0$ are constants, if and only if A satisfies the following properties (A1)–(A3) *and it is maximal with respect to* (A1)–(A3):

(A1) *A is a closed linear operator and $\lambda-A$ is injective for $\lambda>a$;*

(A2) *$D((\lambda-A)^{-m})\supset R(C)$ and $\|(\lambda-A)^{-m}C\|\leq M/(\lambda-a)^m$ for $\lambda>a$ and $m\geq 1$;*

(A3) *$Cx\in D(A)$ and $ACx=CAx$ for $x\in D(A)$.*

Theorem 2. *If A satisfies the equivalent conditions of Theorem* 1, *then the part of A in $\overline{D(A)}$ is the generator of a C_1-semigroup $\{S_1(t); t\geq 0\}$ on $\overline{D(A)}$ satisfying $\|S_1(t)x\|\leq Me^{at}\|x\|$ for $x\in\overline{D(A)}$ and $t\geq 0$, where $C_1=C|_{\overline{D(A)}}$.*

The above-mentioned Arendt's results are the case of $C=I$ (the identity) in Theorems 1 and 2. As direct consequences of Theorems 1 and 2 we have:

Corollary 1. *If A satisfies* (A1)–(A3) *in Theorem* 1 *then $C^{-1}AC$ is the generator of an integrated C-semigroup $\{U(t); t\geq 0\}$ on X satisfying $\|U(t+h)-U(t)\|\leq Mhe^{a(t+h)}$ for $t, h\geq 0$.*

Corollary 2 ([2, Corollary 13.2]). *Suppose $\overline{R(C)}=X$. A is the generator of a C-semigroup $\{S(t); t\geq 0\}$ on X satisfying $\|S(t)\|\leq Me^{at}$ for $t\geq 0$ if and only if A is maximal with respect to* (A2), (A3) *in Theorem* 1 *and* "(A1′) *A is a closed linear operator with $\overline{D(A)}=X$ and λ-A is injective for $\lambda>a$*".

2. Integrated C-semigroups. Let $C\in B(X)$ be injective. A family $\{U(t); t\geq 0\}$ in $B(X)$ is called an *integrated C-semigroup on X*, if

(2.1) $U(\cdot)x: [0, \infty)\to X$ is continuous for $x\in X$,

(2.2) $U(t)x=0$ for all $t>0$ implies $x=0$,

(2.3) there exist $K>0$ and $b\geq 0$ such that $\|U(t)\|\leq Ke^{bt}$ for $t\geq 0$,

(2.4) $U(0)=0$ (the zero operator) and $U(t)C=CU(t)$ for $t>0$,