7. *Parabolic Components of Zeta Functions*

By Nobushige KUROKAWA
Department of Mathematics, Tokyo Institute of Technology

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The functional equation for the Riemann zeta function $\zeta(s)$ was discovered by Euler [1] in 1749 in the form $\zeta(1-s)=\Gamma_{C}(s)\cos(\pi s/2)\zeta(s)$ with $\Gamma_{C}(s)=2(2\pi)^{-s}\Gamma(s)$. Later, Riemann [2] found the symmetric functional equation: $\Gamma_{R}(s)\zeta(s)=\Gamma_{R}(1-s)\zeta(1-s)$ where $\Gamma_{R}(s)=\pi^{-s/2}\Gamma(s/2)$. These two functional equations are equivalent since $\Gamma_{R}(s)\Gamma_{R}(s+1)=\Gamma_{C}(s)$ and $\Gamma_{R}(1+s)\Gamma_{R}(1-s)=(\cos(\pi s/2))^{-1}$, but as is well-known the Riemann's form has been more suggestive to later developments of arithmetic zeta functions containing the adelic view point, where $\Gamma_{R}(s)\zeta(s)$ is considered as the product of local zeta functions.

The same is true for Selberg zeta functions. Let M be a compact Riemann surface of genus $g\geqq 2$, and $\Gamma=\pi_1(M)$ the fundamental group embedded in $\mathrm{PSL}_2(\boldsymbol{R})$, so $M=\Gamma\backslash H$ for the upper half plane H. The zeta function $Z_{\mathrm{hyp}}(s)$ of Γ (or M) is defined by Selberg [3] as the product over all primitive hyperbolic conjugacy classes of Γ. The functional equation of Selberg was not symmetric corresponding to Euler. Later, Vignéras [4] as Riemann presented the symmetric functional equation $Z_{\mathrm{id}}(s)Z_{\mathrm{hyp}}(s)=Z_{\mathrm{id}}(1-s)Z_{\mathrm{hyp}}(1-s)$ with the identity factor $Z_{\mathrm{id}}(s)=\Gamma_2^{C}(s)^{2g-2}=((2\pi)^s\Gamma_2(s)^2\Gamma(s)^{-1})^{2g-2}$ where $\Gamma_2(s)$ is the double gamma function of Barnes. Recently, Voros [5] and Sarnak [6] give the determinant expression

$$Z_{\mathrm{id}}(s)Z_{\mathrm{hyp}}(s)=\det(\Delta-s(1-s))\exp((2g-2)(C+s(1-s)))$$

where Δ is the Laplace operator acting on $L^2(M)$ and $C=-1/4-(1/2)\log(2\pi)+2\zeta'(-1)$. Letting $s\to 1$, they reprove

$$Z'_{\mathrm{hyp}}(1)=\det{}'(\Delta)\exp((2g-2)(C+\log(2\pi)))$$

which was previously shown by string physicists.

We study the case of non-co-compact Γ (non-compact M). The basic case is $\Gamma=\mathrm{PSL}_2(\boldsymbol{Z})$, and hereafter we treat this case since the general feature appears explicitly here. The case of congruence subgroups is quite similar and our method is directly applicable. According to Vignéras [4] we have the symmetric functional equation

$$Z_{\mathrm{hyp}}(s)Z_{\mathrm{id}}(s)Z_{\mathrm{ell}}(s)Z_{\mathrm{par}}(s)=Z_{\mathrm{hyp}}(1-s)Z_{\mathrm{id}}(1-s)Z_{\mathrm{ell}}(1-s)Z_{\mathrm{par}}(1-s)$$

with $Z_{\mathrm{id}}(s)=\Gamma_2^{C}(s)^{1/6}$. Unfortunately $Z_{\mathrm{ell}}(s)$ and $Z_{\mathrm{par}}(s)$ are incompletely (or erroneously) defined in [4]. In the remarkable paper [7], Fischer gives correctly

$$Z_{\mathrm{ell}}(s)=\Gamma(s/2)^{-1/2}\Gamma((s+1)/2)^{1/2}\Gamma(s/3)^{-2/3}\Gamma((s+2)/3)^{2/3}$$

and $Z_{\mathrm{par}}(s)$ a bit implicitly; we refer to Venkov [8] for related calculations. More precisely we have