35. A Note on the Normal Generation of Ample Line Bundles on Abelian Varieties

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Let k be an algebraically closed field, let A be an abelian variety defined over k and let L be an ample line bundle on A. It is well known that $L^{\otimes n}$ is normally generated if $n \geqq 3$ (see Koizumi [2] or Sekiguchi [5], [6]). But $L^{\otimes 2}$ is not normally generated in general because $L^{\otimes 2}$ is not very ample in general. For the very ampleness of $L^{\otimes 2}$, the following result is obtained (see Ohbuchi [3]).

Theorem A. *$L^{\otimes 2}$ is not very ample if and only if (A, L) is isomorphic to $(A_1 \times A_2, \mathcal{O}(\Theta_1 \times A_2 + A_1 \times D_2))$ where A_1 and A_2 are abelian varieties with* $\dim(A_1) > 0$ *and Θ_1 is a theta divisor.*

Our purpose is to give a condition for the normal generation of $L^{\otimes 2}$. The result is as follows:

Theorem. *If* $\operatorname{char}(k) \neq 2$ *and L is a symmetric ample line bundle, then $L^{\otimes 2}$ is normally generated if and only if the origine* 0 *of A is not contained in* $\operatorname{Bs}|L\otimes P_\alpha|$ *for any* $\alpha \in \hat{A}_2 = \{\alpha \in \hat{A}\,;\, 2\alpha = 0\}$ *where $\hat{A}$ is the dual abelian variety of A, P is the Poincaré bundle on $A \times \hat{A}$, $P_\alpha = P_{|A\times\{\alpha\}}$ for $\alpha \in \hat{A}$ and* $\operatorname{Bs}|L\otimes P_\alpha|$ *is the set of all base points of $L \otimes P_\alpha$.*

To prove this theorem, we need three lemmas.

Lemma 1. *If* $\operatorname{char}(k) \neq 2$ *and L is a symmetric ample line bundle, then $\xi^*(p_1^*L \otimes p_2^*L) \simeq p_1^*(L^{\otimes 2}) \otimes p_2^*(L^{\otimes 2})$ where $p_i : A \times A \to A$ is the i-th projection $(i=1,2)$ and $\xi : A\times A \to A \times A$ is defined by $\xi(x, y) = (x+y, x-y)$ for all S-valued points x, y where S is a k-scheme.*

Proof. As $\xi^*(p_1^*L\otimes p_2^*L)_{|A\times\{y\}} \simeq T_y^*L \otimes T_{-y}^*L \simeq L^{\otimes 2}$ for any closed point $y \in A$, therefore $\xi^*(p_1^*L \otimes p_2^*L) \otimes (p_1^*(L^{\otimes 2}))^{-1} \simeq p_2^*M$ for some line bundle M on A by See-Saw theorem. Moreover $\xi^*(p_1^*L\otimes p_2^*L)_{|\{0\}\times A} \simeq L \otimes (-1_A)^*L \simeq L^{\otimes 2}$, hence $M \simeq L^{\otimes 2}$.

Lemma 2. *If* $\operatorname{char}(k) \neq 2$ *and L is an ample line bundle, then*

$$\sum_{\alpha \in \hat{A}_2} \Gamma(A, L \otimes P_\alpha) \xrightarrow{2_A^*} \Gamma(A, 2_A^*L)$$

is an isomorphism.

Proof. This is a well known fact (see Mumford [1]).

Lemma 3. *If L is an ample line bundle, then*

$$\Gamma(A, L^{\otimes n}) \otimes \Gamma(A, L^{\otimes m}) \longrightarrow \Gamma(A, L^{\otimes (n+m)})$$

is surjective if $n \geqq 2$, $m \geqq 3$.

Proof. See Koizumi [2] or Sekiguchi [5], [6].