30. A Note on the Abstract Cauchy-Kowalewski Theorem

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The purpose of this note is to give a simplified proof and an extention of the nonlinear Cauchy-Kowalewski theorem established by Ovsjannikov [5], Nirenberg [3], Nishida [4] and Kano-Nishida [2] (Appendice). The formulation is generalized, and we need only the contraction mapping principle in the proofs. (See also [1] Appendix C.)

Let $\{X_\rho\,;0\leq\rho\leq\rho_0\}$ be a Banach scale so that $X_\rho\subset X_{\rho'}$ and $|\ \ |_\rho\geq|\ \ |_{\rho'}$ for any $\rho_0\geq\rho\geq\rho'\geq0$, where $|\ \ |_\rho$ denotes the norm of X_ρ. Consider the equation

(1) $$u(t)=F(t,u(\cdot)),\qquad 0\leq t\leq T.$$

To state the assumptions on F, we introduce some notations. Let $X_{\rho,t}$ be the space of continuous functions $f(s)$ of $s\in[0,t]$ with values in the Banach space X_ρ, which is equipped with the norm

(2) $$|f|_{\rho,t}=\sup_{0\leq s\leq t}|f(s)|_\rho.$$

We also put $X_{\rho,t}(R)=\{f\in X_{\rho,t}\,;|f|_{\rho,t}\leq R\}$.

We state the assumptions on F:

(F.1) There exist constants $R>0$ and $\gamma_0>0$ such that for any $u\in X_{\rho,\tau}(R)$ $F(t,u(\cdot))$ is an $X_{\rho'}$-valued continuous function on $[0,\tau]$ if $0\leq\rho'<\rho\leq\rho_0-\gamma_0\tau$.

(F.2) For $\rho'<\rho(s)\leq\rho\leq\rho_0-\gamma_0\tau$ and $0<\tau\leq T$, F satisfies the following inequality (3) for any $u,v\in X_{\rho,\tau}(R)$:

(3) $$|F(t,u(\cdot))-F(t,v(\cdot))|_{\rho'}\leq\int_0^t C|u(s)-v(s)|_{\rho(s)}/(\rho(s)-\rho')ds,$$

where C is a constant independent of $t,\tau,u,v,\rho,\rho(s)$ or ρ'.

(F.3) For $0<\tau\leq T$ and $\rho\leq\rho_0-\gamma_0\tau$, $F(t,0)$ is continuous in $X_{\rho,\tau}$ and satisfies

(4) $$|F(t,0)|_{\rho_0-\gamma_0 t}\leq R_0<R.$$

For later use we introduce two Banach spaces $Y_{\rho,\gamma}$ and Z_γ of X_ρ-valued continuous functions, by indicating the norms (the range of t being omitted without confusion):

(5) $$\|u\|_{\rho,\gamma}=\sup_{t\geq0}|u(t)|_{\rho-\gamma t},$$

(6) $$\|u\|_\gamma=\sup_{0\leq\gamma t\leq\rho_0-\rho}|u(t)|_\rho\,\varphi(\gamma t/(\rho_0-\rho)),$$

where $\varphi(t)=(1-t)e^{-t}$. By $Y_{\rho,\gamma}(R)$ we denote the subset $\{f\in Y_{\rho,\gamma}\,;\|f\|_{\rho,\gamma}\leq R\}$. Clearly we have the following:

(7) $\varphi(t)$ is monotone decreasing in $[0,1]$,

(8) $1-\varphi(t)>t$ for $0<t<1$,