98. *The Existence of Solvable Operators in the Ideal*

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§1. **Introduction and preparation.** Let R be the set of all real numbers, and let f be a function on the plane $R\times R$ taking values in R. Let (x, y) be coordinates in $R\times R$. In this note we announce a property and applications of the following polygonal functional equation for all $(x, y) \in R\times R$, $t\in R$:

$$(P)\quad \begin{aligned} &f(x+t,y)+f(x-t,y)+f(x,y+t)+f(x,y-t)\\ &=f(x+t,y+t)+f(x-t,y+t)+f(x+t,y-t)+f(x-t,y-t). \end{aligned}$$

Let F denote the set of all functions $f: R\times R\to R$. Denote by A the algebra of all linear operators on F. As usual, the multiplication in F is composition of operators. For $t\in R$, $f\in F$, define the shift operators $X^t, Y^t: F\to F$ by $(X^t f)(x,y)=f(x+t,y)$ and $(Y^t f)(x,y)=f(x,y+t)$ for all $x, y\in R$. $S\subset A$ denotes the commutative sub-algebra with unit $(1=X^0=Y^0)$ generated by finite linear combinations of shift operators X^t and Y^t on F. Define the operators $\sigma(t)$ and $\theta(t)$ by $\sigma(t)=X^t+X^{-t}+Y^t+Y^{-t}$ and $\theta(t)=(X^t+X^{-t})(Y^t+Y^{-t})=X^tY^t+X^{-t}Y^t+X^tY^{-t}+X^{-t}Y^{-t}$ for $t\in R$. Then the operator $\mu(t)=\sigma(t)-\theta(t)$ for $t\in R$ is in S. For $f\in F$ the operator equation $(\mu(t)f)(x,y)=0$ reduces to polygonal functional equation (P). It is clear that if f satisfies the equation $(\mu(t)f)(x,y)=0$ for $t\in R$, then f also satisfies the equation $(\nu(t)\mu(t)f)(x,y)=0$ for any $\nu(t)\in S$. Hence we consider the ideal generated in S by the family of operators

$$(Q)\qquad \{\mu(t)\mid t\in R\}.$$

In [1] J. Aczél, H. Haruki, M. A. McKiernan and G. N. Sakovič considered the equivalence of polygonal functional equation (P), or briefly, $\sigma(t)-\theta(t)=0$, and the Haruki square mean value equation ([10], [13])

$$(H)\quad f(x+t,y+t)+f(x-t,y+t)+f(x+t,y-t)+f(x-t,y-t)=4f(x,y)$$

or, simply, $\theta(t)-4=0$ under the assumption that $f: R\times R\to R$ is continuous everywhere; it is not necessarily true without continuity assumption. Further, they proved in [1] that if f is continuous and satisfies (H) for all $(x,y)\in R\times R$, $t\in R$, then f is C^∞ on the plane. Hence the only continuous solution of (H) is a certain harmonic polynomial of bounded degree. So, by the equivalency of (P) and (H), the only continuous solution of (P) is also given by the same harmonic polynomial. Moreover, they also obtained [1] the general solution of (H) when no regularity assumptions are imposed on f. On the other hand, if the general theorem of McKiernan [11, Theorem 2, p. 32] is directly applied to equation (H), then one can readily obtain operators corresponding to an equation which we can solve so that