5. On Coprime Integral Solutions of $y^2=x^3+k$

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1. Consider an elliptic curve

$$E(k): y^2=x^3+k \tag{1}$$

with $k \in \boldsymbol{Z}$. The number of integral solutions of the diophantine equation (1), i.e. the number of points $P=(x, y)$ $(x, y \in \boldsymbol{Z})$ on $E(k)$, which is well-known to be finite, will be denoted by $N(k)$, and the number of coprime solutions by $N'(k)$. The value of $\limsup_{k\to\infty} N'(k)$, which will be denoted by c for simplicity's sake, has been studied by Stephens [7], Mohanty and Ramasamy [3], [5]. After $c \geqq 6$ was proved in [3], $c \geqq 8$, $c \geqq 12$ were proved in [7] and [5]. In the next paragraph § 2, we shall improve these results to $c \geqq 20$.

Integral solutions (x_1, y_1), (x_2, y_2), (x_3, y_3) of (1) with $y_1-y_2=y_2-y_3=1$ are called *consecutive*. Mohanty [4] gave four series of such solutions for certain k and asked if there are still other solutions. In § 3, we shall give an affirmative answer to this question.

We recall that the rational points on $E(k)$ form an abelian group with respect to a well-known addition (cf. [1], p. 11).

2. We begin with the following simple lemma.

Lemma 1. *Let* $k=(f^6+g^6+h^6-2f^3g^3-2g^3h^3-2h^3f^3)/4$, *then the following three points* $P_i=(x_i, y_i)$ $(i=1,2,3)$ *are on* $E(k)$.

$$\begin{aligned} x_1&=fg & y_1&=(f^3+g^3-h^3)/2\\ x_2&=fh & y_2&=(f^3-g^3+h^3)/2\\ x_3&=gh & y_3&=(-f^3+g^3+h^3)/2. \end{aligned}$$

We shall omit the straightforward proof.

Remark. Let $f, g, h \in \boldsymbol{Z}$. Then $k \in \boldsymbol{Z}$ if one of f, g, h is even and two others are odd, and P_i are integral (i.e. $x_i, y_i \in \boldsymbol{Z}$, $i=1,2,3$).

Now let $a, b, c \in \boldsymbol{Z}$, $a\equiv d\equiv 0 \pmod 2$, $b\equiv c\equiv 1 \pmod 2$ and put $P_i=(x_i, y_i)$ $(i=1, \cdots, 6)$ where

$$\begin{aligned} x_1&=ab & y_1&=(a^3+b^3-c^3)/2\\ x_2&=ac & y_2&=(a^3-b^3+c^3)/2\\ x_3&=bc & y_3&=(-a^3+b^3+c^3)/2\\ x_4&=bd & y_4&=(-d^3-b^3+c^3)/2\\ x_5&=cd & y_5&=(-d^3+b^3-c^3)/2\\ x_6&=bc & y_6&=(d^3-b^3-c^3)/2. \end{aligned} \tag{2}$$

Then by our Lemma 1, P_1, P_2, P_3 are on $E(k)$ and P_4, P_5, P_6 on $E(k')$ where

$$k=(a^6+b^6+c^6-2a^3b^3-2b^3c^3-2c^3a^3)/4,$$
$$k'=(b^6+c^6+d^6-2b^3c^3-2c^3d^3-2d^3b^3)/4.$$