117. A Note on the Approximate Functional Equation for $\zeta^2(s)$. III

By Yoichi MOTOHASHI
Department of Mathematics, College of Science and Technology, Nihon University, Tokyo

(Communicated by Kunihiko KODAIRA, M. J. A., Dec. 12, 1986)

1. Let $\mathcal{E}_2(s, \alpha t/2\pi)$ be the error-term in the approximate functional equation for $\zeta^2(s)$, i.e.

$$\mathcal{E}_2(s, \alpha t/2\pi)=\zeta^2(s)-\sum_{n\leqq \alpha t/2\pi}{}' d(n)n^{-s}-\chi^2(s)\sum_{n\leqq t/2\pi\alpha}{}' d(n)n^{s-1},$$

where $\chi(s)$ is the Γ-factor in the functional equation for $\zeta(s)$, and the prime indicates that $d(\alpha t/2\pi)$ and $d(t/2\pi\alpha)$ are halved; naturally we use the convention that $d(x)=0$ if x is not an integer.

The problem of finding an asymptotic expansion for $\mathcal{E}_2(s, \alpha t/2\pi)$ has been solved in our former note [2] when $\alpha=1$ the symmetric case. Here we shall show a solution for the non-symmetric case where α is a rational number with a 'not-too-large' denominator. To state our result we introduce some notations: Let $(k, l)=1$, and

$$\Delta(x, l/k)=\sum_{n\leqq x}{}' d(n)\exp(2\pi iln/k)-\frac{x}{k}\left(\log\frac{x}{k^2}+2\gamma-1\right)-E(0, l/k),$$

where γ is the Euler constant, and $E(0, l/k)$ is the value at $s=0$ of the analytic continuation of

$$E(s, l/k)=\sum_{n=1}^{\infty} d(n)\exp(2\pi iln/k)n^{-s}.$$

We put

$$\begin{aligned}Y(s, l/k)=&-\exp(\pi i/4)(2\pi/t)^{1/2}(l/k)^{1-s}\Delta(lt/2\pi k, l/k)\\&+\frac{1}{2\sqrt{\pi}}\exp(\pi i/4)(l/k)^{1/2-s}(kl/2\pi t)^{1/4}\sum_{n=1}^{\infty} d(n)\\&\times\exp(-2\pi i\bar{l}n/k)h(n/kl)n^{-3/4},\end{aligned}$$

where $l\bar{l}\equiv 1 \pmod{k}$ and

$$h(x)=\int_0^{\infty}\exp(-i\pi x\xi)(\xi+1)^{-3/2}d\xi.$$

Theorem. *Let $(k, l)=1$, $l<k$, $kl\leqq t(\log t)^{-20}$. Then we have, for $0\leqq\sigma\leqq 1$,*

$$\chi(1-s)\mathcal{E}_2(s, lt/2\pi k)=Y(s, l/k)+\overline{Y(1-\bar{s}, k/l)}+O((l/k)^{1/2-\sigma}(kl/t)^{1/2}(\log t)^3).$$

Remarks. As has been observed by Jutila ([1, p. 105]), $\mathcal{E}_2(s, \alpha t/2\pi)=\Omega(\log t)$ when α is very close to 1 (e.g. $\alpha=1-ct^{-1/2}$). Thus, if $kl\gg t$ then $\mathcal{E}_2(s, lt/2\pi k)$ cannot be small in general. But our result implies that if kl is relatively small then the approximation becomes significant. This reminds us of the 'major-arc, minor-arc' situation in the theory of trigonometrical method. It should be noted also that the O-term in our theorem