85. On the Existence of Polyharmonic Functions in Lebesgue Classes

By Yoshihiro MIZUTA
Faculty of Integrated Arts and Sciences, Hiroshima University

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For a domain D in the n-dimensional euclidean space R^n, we denote by $H_m(D)$ the class of all functions polyharmonic of order m in D. We say that a subspace $\mathcal{H}$ of $H_m(D)$ is non-trivial if it has an element which is not extended to a function polyharmonic of order m in R^n. Our aim in this note is to find a condition for $\mathcal{H}$ to be non-trivial. This problem is closely related to the removable singularities for polyharmonic functions (see Adams-Polking [1], Harvey-Polking [3], [4], Maz'ja-Havin [5], Mizuta [7]).

The Bessel capacity of index (α, p) of a set E is defined by

$$B_{\alpha,p}(E)=\inf \int |f(y)|^p \, dy,$$

where the infimum is taken over all nonnegative measurable functions f such that $g_\alpha * f(x) \geqq 1$ for all $x \in E$, g_α being the Bessel kernel of order α (cf. Meyers [6]).

Theorem 1. *Let m be a positive integer, $1<p<\infty$, $1/p+1/q=1$ and D be a domain in R^n.*

(i) *If $B_{2m,p}(R^n-D)=0$, then $H_m(D) \cap L^q(D)=\{0\}$.*

(ii) *If $2mp \leqq n$ and $B_{2m,p}(R^n-D)>0$, then $H_m(D) \cap L^q(D)$ is non-trivial.*

(iii) *If $2m-n/p$ is a positive number which is not an integer and R^n-D contains at least two points, then $H_m(D) \cap L^q(D)$ is non-trivial.*

(iv) *If $2m-n/p$ is a positive integer and R^n-D contains three distinct points x_1, x_2, x_3 such that $2x_2=x_1+x_3$, then $H_m(D) \cap L^q(D)$ is non-trivial.*

Proof. Statement (i) is an easy consequence of [1; Theorem B] and the fact that $H_m(R^n) \cap L^q(R^n)=\{0\}$.

Assume that the conditions in (ii) are satisfied. Then we can find mutually disjoint compact subsets $K_1, K_2 \subset R^n-D$ such that $B_{2m,p}(K_i)>0$ for $i=1,2$. By [6; Theorem 16], there exist nonnegative measures μ_1, μ_2 such that the support of μ_i is included in K_i, $\mu_i(K_i)=1$ and $g_{2m}*\mu_i \in L^q(R^n)$ for each i. Consider the function

$$u(x)=\int |x-y|^{2m-n} \, d\mu_1(y)-\int |x-z|^{2m-n} \, d\mu_2(z).$$

Since $g_m*\mu_i \in L^q(R^n)$, $u \in L^q_{\mathrm{loc}}(R^n)$. Further, noting that $u(x)=O(|x|^{2m-n-1})$ as $|x| \to \infty$, we can prove that $u \in L^q(R^n)$. Clearly, $u \in H_m(D)$ and u is not extended to a function polyharmonic of order m in R^n. Thus assertion (ii) is proved.