73. *On Matukuma's Equation and Related Topics*

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§1. **Introduction.** In 1930, Matukuma, an astrophysicist, proposed the following mathematical model to describe the dynamics of a globular cluster of stars,

(M) $$\Delta u+(1+|x|^2)^{-1}u^p=0, \qquad x\in \boldsymbol{R}^3,$$

where $p>1$, u represents the gravitational potential (therefore $u>0$), $\rho=-(4\pi)^{-1}\Delta u=\{4\pi(1+|x|^2)\}^{-1}u^p$ represents the density and $\iiint \rho dx$ represents the total mass (for details, see [1]). Since the globular cluster has the radial symmetry, positive radial entire solutions of (M) (i.e. solutions of (M) with $u(x)=u(|x|)>0$ for all $x\in \boldsymbol{R}^3$) are of particular interest, and the equation (M) reduces to an ordinary differential equation

(M_α) $$u_{rr}+(2/r)u_r+(1+r^2)^{-1}u^p=0\ (r>0), \quad u(0)=\alpha, \quad u_r(0)=0,$$

where $\alpha>0$. For each $\alpha>0$, we denote the global unique solution of (M_α) by $u=u(r;\alpha)$. Studying the structure of solutions of (M_α), Matukuma conjectured:

(i) if $p<3$, then $u(r;\alpha)$ has a finite zero for every $\alpha>0$,

(ii) if $p=3$, then $u(r;\alpha)$ is a positive entire solution with finite total mass for every $\alpha>0$,

(iii) if $p>3$, then $u(r;\alpha)$ is a positive entire solution with infinite total mass for every $\alpha>0$.

In 1938, Matukuma found an interesting exact solution ([2])

(S) $$u(r;\sqrt{3})=\{3/(1+r^2)\}^{1/2} \qquad (p=3),$$

which confirms part of his conjecture.

It turns out that the equation (M_α) is more delicate than Matukuma had expected. In answer to his conjecture, we prove that

(i) if $1<p<5$, then $u(r;\alpha)$ has a finite zero for every sufficiently large $\alpha>0$,

(ii) if $1<p<5$, then $u(r;\alpha)$ is a positive entire solution with infinite total mass for every sufficiently small $\alpha>0$,

(iii) if $p\geq 5$, then $u(r;\alpha)$ is a positive entire solution with infinite total mass for every $\alpha>0$.

The conclusions above follow from our more general results stated in Section 2 below. (Set $K(r)=1/(1+r^2)$, $n=3$, and $\sigma=0$ in Theorem 4, $l=-2$ and $C=1$ in Theorem 3, and $\sigma=0$ in Theorem 5.) It is rather interesting

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