71. Estimation of Multiple Laplace Transforms of Convex Functions with an Application to Analytic (C_0)-semigroups

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1. This note is concerned with a new method of estimating multiple Laplace transforms of convex functions of the form

$$\int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^n \lambda_i \xi_i\right) f\left(\sum_{i=1}^n \xi_i\right) d\xi_1 \cdots d\xi_n, \tag{1}$$

where $\lambda_i > 0$ for $i=1, \cdots, n$ and $f(\xi)$ is a nonnegative convex function on $(0, \infty)$.

This problem arose from estimating the iteration of resolvents of the infinitesimal generator A of an analytic (C_0)-semigroup $\mathcal{T} = \{T(t) : t \geqq 0\}$ on a Banach space X. Consider the operators

$$A_\theta \prod_{i=1}^n (I - h_i A)^{-1} \tag{2}$$

for $h_i > 0$, $i=1, \cdots, n$ and $n=1, 2, \cdots$, where we assume that $\|T(t)\| \leqq Me^{-\omega t}$ for $t \geqq 0$ and some $M \geqq 1$ and $\omega > 0$; $\theta \in (0, 1)$; $A_\theta = -(-A)^\theta$; and $(-A)^\theta$ is the fractional power of $-A$. By means of the relation

$$(I - h_i A)^{-1} x = h^{-1} \int_0^\infty e^{-(\xi/h)} T(\xi) x \, d\xi, \qquad x \in X,$$

$A_\theta \prod_{i=1}^n (I - h_i A)^{-1} x$ is written as

$$\left(\prod_{i=1}^n h_i^{-1}\right) \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^n h_i^{-1} \xi_i\right) A_\theta T\left(\sum_{i=1}^n \xi_i\right) x \, d\xi_1 \cdots d\xi_n.$$

Since $\|A_\theta T(\xi)\|$ is dominated pointwise by the convex function $f(\xi) \equiv c_\theta \xi^{-\theta}$ on $(0, \infty)$, c_θ being a positive constant depending only upon θ, the norm of the operator (2) is bounded above by the following type of multiple integral:

$$\left(\prod_{i=1}^n h_i^{-1}\right) \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^n h_i^{-1} \xi_i\right) f\left(\sum_{i=1}^n \xi_i\right) d\xi_1 \cdots d\xi_n. \tag{3}$$

Our objective here is to describe a new method for estimating the above multiple integrals and show that they are bounded by the value of the integral

$$(m-1)!^{-1} h^{-m} \int_0^\infty \xi^{m-1} e^{-(\xi/h)} f(\xi) \, d\xi, \tag{4}$$

provided that $n \geqq m$, $h = m^{-1} \sum_{i=1}^n h_i$ and $h_i \leqq h$ for $i=1, \cdots, n$.

Let m be any positive integer. Let $m-1 \leqq \alpha < m$ and consider the function $f(\xi) = c_\alpha \xi^{-\alpha}$ on $(0, \infty)$, where c_α is a positive constant. Then $\int_0^\infty \xi^{m-1} e^{-(\xi/h)} f(\xi) d\xi < \infty$ and the integral (4) with this singular convex function is evaluated as $(m-1)!^{-1} c_\alpha \Gamma(m-\alpha) h^\alpha$, where $\Gamma(s)$ denotes the gamma