

### 59. On Semi-Idempotents in Rings

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An element  $\alpha \neq 0$  of a ring  $R$  is called *semi-idempotent* if and only if  $\alpha$  is not in the proper two sided ideal generated by  $\alpha^2 - \alpha$ ,  
i.e.,  $\alpha \notin R(\alpha^2 - \alpha)R$  or  $R(\alpha^2 - \alpha)R = R$ .  
0 is also counted among semi-idempotents. Obviously idempotents and units of  $R$  are semi-idempotents.

In [1] W. B. Vasantha proved certain results about semi-idempotents in group rings. We give a generalization of one of the results in the paper and use it to give a characterization of local rings.

Throughout the rest of this paper,  $R$  will denote a ring with identity.  $\text{Rad } R$  denotes the Jacobson radical of  $R$ .

**Theorem A.** *If  $\alpha \in R$ , then either  $\alpha$  is semi-idempotent or  $R(1 - \alpha)R = R$ .*

*Proof.* If  $\alpha$  is not semi-idempotent, we have

$$\alpha \in R(\alpha^2 - \alpha)R \subseteq R(1 - \alpha)R.$$

Also  $(1 - \alpha) \in R(1 - \alpha)R$ . So we have  $R(1 - \alpha)R = R$ .

**Remark.** This was proved in [1] for the case  $R = KG$ ,  $K$  a field,  $G$  abelian.

**Lemma 1.** *Non-zero elements of  $\text{Rad } R$  are not semi-idempotent.*

*Proof.* Let  $\alpha$  be a non-zero element of  $\text{Rad } R$ . As  $(1 - \alpha)$  is invertible,  $R(\alpha^2 - \alpha)R = R\alpha R (\subseteq \text{Rad } R)$  is therefore a proper ideal containing  $\alpha$ . Hence  $\alpha$  is not semi-idempotent.

**Theorem B.** *The following are equivalent for a ring  $R$ .*

- (1)  $(R/\text{Rad } R)$  is a division ring.
- (2) The only semi-idempotents of  $R$  are units and zero.

*Proof.* Suppose that the only semi-idempotents of  $R$  are units and zero. Consider

$I = \{\alpha \in R \mid R\alpha R \neq R\}$ . If we show that  $I$  is closed under addition, then  $I$  will be a two sided ideal.

Let  $\alpha, \beta \in I$ . If  $\alpha + \beta \notin I$ , we have  $R(\alpha + \beta)R = R$ , i.e., there exists elements  $a, b, a \in R\alpha R, b \in R\beta R$ , such that  $a + b = 1$ .

Neither  $a$  nor  $b$  can be zero. But as  $a \in R\alpha R, R\alpha R \neq R$ ,  $a$  is not semi-idempotent by hypothesis. Hence  $R(1 - a)R = R$  by Theorem A. That is  $RbR = R$ , which contradicts the fact that  $b \in I$ .

Hence  $I$  is closed under addition. It is easily seen that  $I$  is the unique maximal two sided ideal. Now we claim that it is actually a unique maximal left ideal. If  $\alpha$  is any non-zero element such that  $R\alpha \neq R$ , then  $\alpha$  is not