56. *Discrepancy with respect to Weighted Means of Some Sequences*

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1. It is well-known that for irrational α's of small type the sequences $(n\alpha)$, $n=1, 2, \cdots$, have uniformly low discrepancies [1 : pp. 121–126]. In this note we shall show the connection between the type of α and the P-discrepancy of the sequence $(a_n\alpha)$, $n=1, 2, \cdots$, where (a_n) is a non-decreasing sequence of integers and the P-discrepancy is a generalized notion of discrepancy. Furthermore, we shall give a quantitative form of Theorem 8 of Tsuji [4] with respect to weighted uniform distribution mod 1. This result contains Theorem 4.1 of Niederreiter [3], Satz 3 of Niederreiter and Tichy [2] and Satz 3 of Tichy [5] as special cases.

2. Definition 1. Let $P=(p_n)$, $n=1, 2, \cdots$, be a sequence of non-negative real numbers with $p_1>0$. For $N\geqq 1$, put $s_N=p_1+p_2+\cdots+p_N$. Given a sequence $\omega=(x_n)$, $n=1, 2, \cdots$, of real numbers and a positive integer N, the P-discrepancy (mod 1) of the first N terms of ω is defined by

$$D_N(P\,;\,\omega)=\sup_I\left|(1/s_N)\sum_{n=1}^{N}p_n c_I(\{x_n\})-|I|\right|,$$

where the supremum is taken over all intervals I in $[0, 1)$, c_I is the characteristic function of I, $\{x_n\}$ is the fractional part of x_n, and $|I|$ is the length of I.

Definition 2. An irrational number α is said to be of constant type if there exists a constant $C>0$ such that for all integers $q>0$, $q\|q\alpha\|\geqq C$ holds, where $\|t\|=\min_{n\in \boldsymbol{Z}}|t-n|$ for $t\in \boldsymbol{R}$.

Definition 3. Let η be a positive real number or infinity. An irrational number α is said to be of type η if η is the supremum of all γ for which $\underline{\lim}_{q\to\infty}\, q^{\gamma}\|q\alpha\|=0$, where q runs through positive integers.

3. Results. Let $p(t)\in C^1[1, \infty)$ be a positive, non-increasing function. We put $p_n=p(n)$ for $n=1, 2, \cdots$. We assume throughout that $\lim_{N\to\infty} s_N=\infty$. Putting $s(t)=\int_1^t p(u)du$ for $t\geqq 1$, the partial sum s_N is asymptotically equal to $s(N)$ as $N\to\infty$.

Theorem 1. *Let $g(t)\in C^2[1, \infty)$ be a positive, strictly increasing function satisfying the following conditions:*

(1) $g(t)\to\infty$ *as* $t\to\infty$,

(2) $g'(t)\to$ *constant* <1 *monotonically as* $t\to\infty$,

(3) $g'(t)/p(t)$ *is monotone for* $t\geqq 1$.

Then for $P=(p(n))$ and $\omega=([g(n)]\alpha)$ with α irrational, there exists an absolute constant C such that