# 46. Continuity Theorem for Non-linear Integral Functionals and Aumann-Perles' Variational Problem

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**1. Introduction.** Let $(T, \mathcal{E}, \mu)$ be a measure space and assume that a couple of functions $u: T\times \boldsymbol{R}^l \to \boldsymbol{R}$ and $g: T\times \boldsymbol{R}^l \to \boldsymbol{R}^k$, as well as a vector $\omega \in \boldsymbol{R}^k$ are given. Consider the well-known Aumann-Perles' variational problem formulated as follows:

$$\text{(P)}\qquad \begin{cases} \underset{x}{Maximize} \int_T u(t, x(t))d\mu \\ \quad subject\ to \\ \qquad \int_T g(t, x(t))d\mu \leqq \omega. \end{cases}$$

The existence of optimal solutions for (P) has been investigated by Artstein [2], Aumann-Perles [3], Berliocchi-Lasry [5], Maruyama [8] and others. In this paper, we shall present an alternative approach to the existence problem, being based upon the continuity theorem for non-linear integral functionals due to Berkovitz [4] and Ioffe [6].

**2. Continuity and compactness of level sets for non-linear integral functionals.** In the proof of our main theorem discussed in the next section, we shall effectively make use of a couple of results in non-linear functional analysis. We had better summarize them here for the sake of readers' convenience.

**Continuity Theorem** (Berkovitz [4], Ioffe [6]). *Let $(T, \mathcal{E}, \mu)$ be a non-atomic complete finite measure space and $f: T\times \boldsymbol{R}^l \times \boldsymbol{R}^k \to \bar{\boldsymbol{R}}$ be a convex normal integrand. Define a non-linear functional $J: L^p(T, \boldsymbol{R}^l)\times L^q(T, \boldsymbol{R}^k) \to \bar{\boldsymbol{R}} (p, q \geqq 1)$ by*

$$J(x, y)=\int_T f(t, x(t), y(t))d\mu.$$

*If there exist some $a \in L^{q'}(T, \boldsymbol{R}^k)$ (where $1/q+1/q'=1$) and $b \in L^1(T, \boldsymbol{R})$ such that*

$$f(t, x, y) \geqq \langle a(t), y\rangle + b(t)$$
$$(\langle \cdot, \cdot\rangle \text{ stands for the inner product})$$

*for all $(t, x, y) \in T\times \boldsymbol{R}^l \times \boldsymbol{R}^k$, then $J$ is sequentially lower semi-continuous with respect to the strong topology on $L^p(T, \boldsymbol{R}^l)$ and the weak topology on $L^q(T, \boldsymbol{R}^l)$.*

**Compactness Theorem** (Ioffe-Tihomirov [7]). *Let $(T, \mathcal{E}, \mu)$ be a finite measure space and $f: T\times \boldsymbol{R}^l \to \bar{\boldsymbol{R}}$ be $(\mathcal{E}\otimes\mathcal{B}(\boldsymbol{R}^l), \mathcal{B}(\bar{\boldsymbol{R}}))$-measurable, where $\mathcal{B}(\cdot)$ stands for the Borel $\sigma$-field on $(\cdot)$. If $f$ satisfies the growth condition:*