95. On the Compactness Criterion for Probability Measures on Banach Spaces

By Jun KAWABE
Department of Information Sciences, Tokyo Institute of Technology

(Communicated by Kôsaku YOSIDA, M. J. A., Dec. 12, 1985)

1. Introduction. A compactness criterion for a set of probability measures on a real separable Hilbert space was given by Prokhorov [11, Theorem 1.14], in terms of their characteristic functionals. In this note we shall prove that a natural generalization of Prokhorov's result to Banach spaces is not valid unless X is isomorphic to a Hilbert space. This is also concerned with author's paper [7].

The author wishes to express his hearty thanks to Professor Hisaharu Umegaki for many kind suggestions and advice.

2. Preliminaries. Let X be a real separable Banach space, X^* its topological dual space and $\mathcal{B}(X)$ the Borel σ-algebra. By a *random element* in X defined on a basic probability space $(\Omega, \mathcal{A}, P)$ we mean a measurable mapping $(\Omega, \mathcal{A}, P)\to(X, \mathcal{B}(X))$. Every random element ξ induces on $(X, \mathcal{B}(X))$ the probability measure $\mu_\xi=P\circ\xi^{-1}$ which is called its *distribution*. A random element ξ is said to be *Gaussian* if for each $f\in X^*$, $\langle\xi(\cdot), f\rangle$ is a (possibly degenerate) real Gaussian random variables on $(\Omega, \mathcal{A}, P)$.

We identify the set $\mathcal{P}(X)$ of all probability measures on $(X, \mathcal{B}(X))$ with the corresponding subset of $C(X)^*$ under the natural injection $\mu\in\mathcal{P}(X)\to\int_X \varphi(x)\mu(dx)$, $\varphi\in C(X)$, where $C(X)$ is the Banach space of all bounded continuous real functions on X. In this note we define the topology on $\mathcal{P}(X)$ as the relative topology induced by the weak* topology on $C(X)^*$. Then $\mathcal{P}(X)$ is a Polish space (see [11]). For each $\mu\in\mathcal{P}(X)$ the *characteristic functional* of μ is defined by

$$\hat{\mu}(f)=\int_X \exp\{i\langle x, f\rangle\}\mu(dx), \qquad f\in X^*.$$

We shall denote by $\mathcal{N}(X^*, X)$ the Banach space of all nuclear operators from X^* into X with the nuclear norm $\nu(\cdot)$ (see [4] and [12]). A nuclear operator $R: X^*\to X$ is called an *S-operator* if it is positive and symmetric, i.e., $\langle Rf, f\rangle\geq 0$ for all $f\in X^*$ and $\langle Rf, g\rangle=\langle Rg, f\rangle$ for all $f, g\in X^*$. Let ξ be a random element in X satisfying $\int\|\xi(\omega)\|^2P(d\omega)<\infty$. Then the operator $R_\xi: X^*\to X$ defined by the equality

$$R_\xi f=\int_\Omega \langle\xi(\omega), f\rangle\xi(\omega)P(d\omega)$$

(the integral is understood in the sense of Bochner) is an S-operator, and it is called the *covariance operator* of ξ (see [2]).