94. A Stochastic Differential Equation Arising from the Vortex Problem

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1. Introduction. The purpose of this paper is to solve a stochastic differential equation (SDE) which represents the vortex flow in the *whole plane*.

A system of n vortices $Z_t=(Z_t^1, \cdots, Z_t^n)$ ($Z_t^i \in R^2$ is the position of the i^{th} vortex at time t and $\gamma_i \in R$ its vorticity intensity) in a viscous and incompressible fluid satisfies the following SDE.

$$dZ_t^i = \sigma dB_t^i + \sum_{\substack{j=1\\ j\neq i}}^{n} \gamma_j K(Z_t^i - Z_t^j)dt, \qquad 1 \leq i \leq n, \tag{1}$$

where

$$K(z) = \nabla^\perp G(z) \qquad z=(x, y) \in R^2, \tag{2}$$

$G(z) = -(2\pi)^{-1}\log|z|$, $\nabla^\perp = (\partial/\partial y, -(\partial/\partial x))$, $(B_t^1, \cdots, B_t^n)$ is a $2n$-dim. Brownian motion and σ is a constant which is related to the viscosity. Since the coefficients are singular on the set

$$S = \bigcup_{\substack{i\neq j\\ i,j=1}}^{n} \{(z_i) \in R^{2n}\,;\, z_i = z_j\},$$

it is not easy to solve (1). Let L be the generator of (1):

$$L = \nu\Delta + \sum_{\substack{i\neq j\\ i,j=1}}^{n} \gamma_j(\nabla_i^\perp G(z_i - z_j))\cdot\nabla_i \tag{3}$$

where

$$\nu = \frac{1}{2}\sigma^2, \quad \nabla_i = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right) \quad \text{and} \quad \nabla_i^\perp = \left(\frac{\partial}{\partial y_i}, -\frac{\partial}{\partial x_i}\right).$$

We can rewrite this as

$$L = \nu\Delta + \sum_{\substack{i\neq j\\ i,j=1}}^{n} \gamma_j \nabla_i^\perp \cdot (G(z_i - z_j)\nabla_i). \tag{4}$$

One might expect to apply PDE results by taking advantage of this divergence structure. However, they do not apply to the case considered here, because $G(z_i - z_j)$ has a *log-type* singularity.

The key point of the proof is to observe that L is a differential operator of a *generalized divergence form* defined in Section 2 and apply a result obtained in [3].

The coefficients $K(z_i - z_j)$ are locally Lipschitz continuous on $R^{2n} - S$. Hence (1) is uniquely solvable till Z_t hits S. The problem is to show that Z_t is conservative on $R^{2n} - S$. Now, we state our main theorem.

Theorem. *Let* $\tau = \inf\{t > 0 : Z_t \in S\}$. *Then for any* $x \in R^{2n} - S$,

$$P_x\{\tau < \infty\} = 0. \tag{5}$$