89. A Note on the Mean Value of the Zeta and L-functions. II

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1. In the present note we consider the mean square of *individual* Dirichlet L-functions.

Let χ be a *primitive* character (mod q), and put

$$E(T,\chi)=\int_0^T\left|L\left(\frac{1}{2}+it,\ \chi\right)\right|^2dt-\frac{\varphi(q)}{q}T\left\{\log(qT/2\pi)+2\gamma+2\sum_{p|q}(\log p)/(p-1)\right\},$$

where φ is the Euler function, γ the Euler constant, and p is a prime divisor of q. Then our problem is to find an estimate of $E(T,\chi)$ as uniform as possible for both parameters q and T. Our argument is based on the following χ-analogue of the important formula (3.4) of Atkinson [1].

Lemma 1. *If* $0<Re(u)<1$ *then*

$$(1)\qquad L(u,\chi)L(1-u,\bar\chi)=\frac{\varphi(q)}{q}\left\{\frac{1}{2}\left(\frac{\Gamma'}{\Gamma}(u)+\frac{\Gamma'}{\Gamma}(1-u)\right)+2\gamma+\log\frac{q}{2\pi}+2\sum_{p|q}\frac{\log p}{p-1}\right\}+g(u,\chi)+g(1-u,\bar\chi),$$

where $g(u,\chi)$ *is the analytic continuation of*

$$(2)\qquad \sum_{n=1}^{\infty}a(n,\chi)\int_0^\infty\exp(-2\pi iny/q)y^{-u}(1+y)^{u-1}dy+\sum_{n=1}^{\infty}\overline{a(n,\bar\chi)}\int_0^\infty\exp(2\pi iny/q)y^{-u}(1+y)^{u-1}dy,$$

which is convergent when $Re(u)<0$. *Here*

$$a(n,\chi)=q^{-1}\sum_{a|n}\sum_{m=1}^{q}\chi(m)\bar\chi(m+a)\exp(2\pi imn/aq).$$

This can be proved by a simple modification of our argument used in [6]. We denote by $g_1(u,\chi)$ the first sum of (2). To get an explicit representation of $g_1(u,\chi)$ which holds at least for $Re(u)<3/4$, we need some information on

$$A(x)=\sum_{n\le x}a(n,\chi).$$

To this end we put

$$F(s,\chi)=\sum_{n=1}^{\infty}a(n,\chi)n^{-s},$$

which is obviously convergent for $Re(s)>1$. Expressing $F(s,\chi)$ by a combination of Hurwitz zeta-functions, we get

Lemma 2. $F(s,\chi)$ *is entire, and when* $Re(s)<0$

$$F(s,\chi)=2(q\tau(\chi))^{-1}(2\pi/q)^{2(s-1)}\Gamma^2(1-s)\times\sum_{n=1}^{\infty}\chi(n)d(n)n^{s-1}(\chi(-1)\exp(-2\pi in/q)-\cos(\pi s)\exp(2\pi in/q)),$$