

88. On Some Class Number Relations for Galois Extensions

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Let k be an algebraic number field of finite degree over \mathbf{Q} , k_v be the completion of k at a place v . When v is non-archimedean, we put $v = \mathfrak{p}$ and denote by \mathfrak{o}_v the ring of \mathfrak{p} -adic integers in k_v . We denote by S_∞ the set of all archimedean places of k .

Let T be a torus defined over k . We denote by $T(k)$, $T(k_v)$ the subgroups of T of points rational over k , k_v , respectively. The adèle group of T over k will be written $T(k_A)$. The unique maximal compact subgroup of $T(k_v)$ will be written $T(\mathfrak{o}_v)$. The latter group is described as

$$T(\mathfrak{o}_v) = \{x \in T(k_v); |\xi(x)|_{\mathfrak{p}} = 1 \text{ for all } \xi \in \hat{T}(k_v)\}$$

where $\hat{T} = \text{Hom}(T, G_m)$, the character module of T and $\hat{T}(k_v)$ is the submodule of \hat{T} of characters defined over k_v . Finally, we put

$$T(k_A)_\infty = \prod_{v \in S_\infty} T(k_v) \times \prod_{\mathfrak{p}} T(\mathfrak{o}_v)$$

and define the class number of T over k by

$$h_T = [T(k_A) : T(k)T(k_A)_\infty].$$

Let K be a finite Galois extension of k . If $T = R_{K/k}(G_m)$, the torus obtained from the multiplicative group G_m by the restriction of the field of definition from K to k , then h_T coincides with the usual class number h_K of K . Consider the exact sequence of tori defined over k :

$$0 \longrightarrow R_{K/k}^{(1)}(G_m) \longrightarrow R_{K/k}(G_m) \xrightarrow{N} G_m \longrightarrow 0$$

where N is the norm map for K/k and $R_{K/k}^{(1)}(G_m) = \text{Ker } N$. As mentioned above, tori G_m , $R_{K/k}(G_m)$ have class numbers h_k , h_K , respectively. We shall denote by $h_{K/k}$ the class number of the torus $R_{K/k}^{(1)}(G_m)$. Then one obtains a positive rational number $E(K/k)$ invariantly attached to a Galois extension K/k :

$$E(K/k) \stackrel{\text{def}}{=} \frac{h_K}{h_k h_{K/k}}.$$

The celebrated formula of Gauss ($h_K^+ = h_K^* 2^{t_K - 1}$, K/\mathbf{Q} = a quadratic extension, h_K^+ = the class number of K in the narrow sense, h_K^* = the number of classes in a genus, t_K = the number of rational primes ramified in K/\mathbf{Q}) on the genera of quadratic forms may be described as establishing an equality between $E(K/\mathbf{Q})$ and other arithmetical invariants of K . Our general arithmetic theory of tori, i.e. the theory of isogenies, class number formulas, Tamagawa numbers, etc. furnishes us with general tools to determine $E(K/k)$ for any Galois extension ([1], [2], [4]).

In our Theorem below the following notation is used.