86. On a Problem of R. Brauer on Zeta-Functions of Algebraic Number Fields

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1. Introduction. Let $\zeta_K(s)$ denote the Dedekind zeta-function of an algebraic number field K. It has been shown by R. Brauer [3] that if Ω_1 and Ω_2 are two finite algebraic number fields which are both normal over their intersection k and their compositum is K, then

$$\zeta_K(s)\zeta_k(s)/\zeta_{\Omega_1}(s)\zeta_{\Omega_2}(s)$$

is an entire function. Let K_1 and K_2 be finite algebraic number fields over $k=K_1\cap K_2$. Suppose now that at least one of K_1, K_2 is non-normal over k, and $K=K_1K_2$. Does it happen that also in this case the function $\zeta_K(s)\zeta_k(s)/\zeta_{K_1}(s)\zeta_{K_2}(s)$ becomes an entire function? We call this question R. Brauer's problem, and show that it has positive answer in some cases.

2. Main theorems.

Theorem 1. *$\zeta_{Q(\sqrt[p]{n},\sqrt[p]{m})}(s)\zeta(s)/\zeta_{Q(\sqrt[p]{n})}(s)\zeta_{Q(\sqrt[p]{m})}(s)$ is an entire function of s, where p is an odd prime and n, m are p-free relatively prime rational integers.*

Proof. Let $\zeta=\exp(2\pi i/p)$. Then $\boldsymbol{Q}(\sqrt[p]{n},\zeta)/\boldsymbol{Q}$ is normal and $T=\mathrm{Gal}(\boldsymbol{Q}(\sqrt[p]{n},\zeta)/\boldsymbol{Q})$ is generated by the elements σ, τ as follows $\sigma^p=\tau^{p-1}=e$, $\tau\sigma\tau^{-1}=\sigma^g$, where g is a primitive root mod p and the elements σ and τ are characterized by $\sigma:\zeta\to\zeta$, $\sqrt[p]{n}\to\sqrt[p]{n}\zeta$, $\tau:\zeta\to\zeta^g$, $\sqrt[p]{n}\to\sqrt[p]{n}$. The group T has $p-1$ linear characters (i.e., irreducible characters of degree one) and precisely one simple non-linear character χ_p such that $\chi_p(e)=p-1$. Here $\chi_p(\rho)=-1$ for $\rho\in\langle\sigma\rangle-\{e\}$ and $\chi_p(\rho)=0$ for $\rho\notin\langle\sigma\rangle$. We consider the field $M=\boldsymbol{Q}(\sqrt[p]{n},\sqrt[p]{m},\zeta)$. Let τ^* be the element of $G=\mathrm{Gal}(M/\boldsymbol{Q})$ such that $\tau^*:\zeta\to\zeta^g$, $\sqrt[p]{n}\to\sqrt[p]{n}$, $\sqrt[p]{m}\to\sqrt[p]{m}$. Then $\Omega=\boldsymbol{Q}(\sqrt[p]{n},\sqrt[p]{m})$ is the intermediate field of M over $\boldsymbol{Q}$ fixed by the cyclic subgroup $H=\langle\tau^*\rangle\subset G$ so that $H=\mathrm{Gal}(M/\Omega)$. Next let δ be the element of $\mathrm{Gal}(M/\boldsymbol{Q})$ such that $\delta;\zeta\to\zeta$, $\sqrt[p]{n}\to\sqrt[p]{n}$, $\sqrt[p]{m}\to\sqrt[p]{m}\zeta$. Then $F=\boldsymbol{Q}(\sqrt[p]{n},\zeta)$ is the fixed field of $N=\langle\delta\rangle$ and we have $\mathrm{Gal}(\boldsymbol{Q}(\sqrt[p]{n},\zeta)/\boldsymbol{Q})\cong G/N$. Here we consider the map $G\xrightarrow{\varphi}G/N\xrightarrow{\chi_p}\boldsymbol{C}$. If we denote $\lambda_p(x)=\chi_p(\varphi(x))$, then λ_p is one of the irreducible characters of G. In particular, $\lambda_p(\tau^*)=\chi_p(\tau)=0$. Let 1_H be the principal character of H, and we denote by 1_H^G the induced character of G. $\lambda_p|_H$ denotes the restriction of λ_p to H. Frobenius reciprocity yields

$$(1_H^G,\lambda_p)_G=(1_H,\lambda_p|_H)_H=\frac{1}{p-1}\sum_{h\in H}\lambda_p|_H(h)$$
$$=\frac{1}{p-1}\Big\{\lambda_p|_H(e)+\sum_{e\neq h\in H}\lambda_p|_H(h)\Big\}=\frac{1}{p-1}\{(p-1)+0+0+\cdots+0\}=1.$$