70. On Riemann Type Integral of Functions with Values in a Certain Fréchet Space

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1. Introduction. Lex X be a Fréchet space [1] [5] with quasi-norm $\|\ \|$ such that, for every $x \in X$ and real number a, $\|ax\|=|a|^{\alpha}\|x\|$ holds for some fixed α, $0<\alpha<1$. We want to consider some sort of integrals of functions defined on a bounded closed interval and taking values in this space. But the theory of the Bochner integral does not apply, since X is not a Banach space, nor is the theory of Riemann integrals extended to this case because of slowness of the convergence $\|ax\|\to 0$ as $a\to 0$. In this paper we prove that Riemann type integrals exist for Hölder continuous functions with exponent γ if $\gamma>1-\alpha$, and we give an upper bound of the norm of the integral in terms of γ and Hölder constant. This integral is motivated by the problem of canonical representations of stationary symmetric α-stable processes.

2. Theorems. Let X be a Fréchet space with the property stated above and x_t be a function of $t\in I=[a, b]$ which has values in X. Sometimes we write $x_t=x(t)$.

Definition 1. Let γ, δ_0, K be positive numbers. We call x_t satisfies Condition $C_\gamma(\delta_0, K)$ if $\|x_t-x_s\|\leq K|t-s|^{\gamma}$ whenever $t, s\in I$ and $|t-s|\leq\delta_0$.

Let $\{I_i, 1\leq i\leq n\}$ be a partition of I such that $a=a_0<a_1<\cdots<a_n=b$, $I_i=[a_{i-1}, a_i]$. A pair of $\{I_i\}$ and $\{t_i\}$, $t_i\in I_i$, is denoted by $S=(\{I_i\}, \{t_i\})$. The length of I_i is denoted by $|I_i|$.

Definition 2. Suppose that x_t is a function defined on I. We say that x_t is Riemann type integrable over I if there is an element $\mathcal{J}$ in X with the following property: For each $\varepsilon>0$, there is $\delta>0$ such that

$$\left\|\sum_{i=1}^{n}|I_i|x(t_i)-\mathcal{J}\right\|<\varepsilon$$

whenever $S=(\{I_i\}, \{t_i\})$ satisfies $\max_{1\leq i\leq n}|I_i|<\delta$. We call $\mathcal{J}$ Riemann type integral and write $\mathcal{J}=\int_I x_t dt$.

Then we have the following theorems.

Theorem 1. *If x_t satisfies Condition $C_\gamma(\delta_0, K)$ for some δ_0, K and γ such that $1\geq\gamma>1-\alpha$, then x_t is Riemann type integrable over I.*

Theorem 2. *Under the same conditions as Theorem 1, we have the following inequality:*

$$\left\|\int_I x_t dt\right\|\leq M^{1-\alpha}|I|^{\alpha}\sup_{t\in I}\|x_t\|+M^{-\rho}|I|^{\alpha+\gamma}KA_{\alpha\gamma}$$

where $\rho=\alpha+\gamma-1$, $A_{\alpha\gamma}=2^{1-2\alpha}2^{\rho}/(2^{\rho}-1)+2^{\gamma}$ and M is any number bigger than $2|I|/\delta_0$.