69. *First Hitting Time for Bessel Processes*

By Junji TAKEUCHI
Department of Mathematics, Ochanomizu University

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By a Bessel process with index α ($\alpha>0$), we mean a conservative diffusion process on the half line $[0, \infty)$ determined by the generator

$$A=\frac{1}{2}\left(\frac{d^2}{dx^2}+\frac{\alpha-1}{x}\frac{d}{dx}\right).$$

In the case $0<\alpha<2$, an appropriate boundary condition must be imposed at the origin. In this note we restrict ourselves to the reflecting barrier case.

The following theorem for the d-dimensional Brownian motion is well known. Let σ_r denote the first exit time from the ball B_r with center 0 and radius r. Suppose $\|x\|\leqq r$, where $\|x\|$ denotes the Euclidean norm of x. Then the expect time spent in B_r by Brownian motion starting at x is given by

$$E_x(\sigma_r)=\frac{r^2-\|x\|^2}{d}.$$

The object of this note is to extend this result to the Bessel processes with reflecting barrier, replacing d by general α. Further we will derive explicitly the second moment of the first passage time to the point r.

Let T_r denote the first hitting time of the point r by the Bessel process $X(t)$, that is,

$$T_r=\inf\{t>0: \ X(t)=r\}.$$

Proposition 1. *Consider points $a<x<b$. Then we have*

$$P_x(T_a<T_b)=\begin{cases}\dfrac{x^{2-\alpha}-b^{2-\alpha}}{a^{2-\alpha}-b^{2-\alpha}} & \text{if } \alpha\neq 2\\[2ex] \dfrac{\log b-\log x}{\log b-\log a} & \text{if } \alpha=2.\end{cases}$$

Proof. Let $S(x)$ be a scale function for a regular diffusion on an interval I of the line. Therefore S is a strictly increasing function such that if $a<x<b$ and $a, b\in I^\circ$ (here I° is the interior of I), and the probability for the process reaching a before b is

$$P_x(T_a<T_b)=\frac{S(b)-S(x)}{S(b)-S(a)}.$$

We may take $S(x)=\log x$ if $\alpha=2$, $S(x)=(2-\alpha)^{-1}x^{2-\alpha}$ if $\alpha\neq 2$ and so the desired formula is obtained.

By the standard argument in Markov process (K. Ito [2], F. B. Knight [4]), we obtain