66. A Valuational Interpretation of Kummer's Theory of Ideal Numbers

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It is well-known that Kummer created "ideal complex numbers" in order to save the unique factorization into prime factors ([4], [3]). In modern terminology, they were the valuations in the cyclotomic field: *Ce que fait Kummer revient exactement, en langage moderne, à definir les valuations sur corps cyclotomiques* (Bourbaki [1], p. 122).

In the present paper, we shall show that Kummer's idea can be directly used to prove the fundamental theorem on the extension of valuations, in even simpler way than in usual proofs, if we invoke a conception of Dedekind presented in the supplements to the second edition of Dirichlet's *Vorlesungen über Zahlentheorie* [2].

We mean by a *valuation* a discrete normed exponential valuation. Let K be a field, v a valuation of K, A the valuation ring of v and π a uniformizer of v:

$$A=\{x(\in K)\,|\,v(x)\geqq 0\}, \quad \text{and} \quad v(\pi)=1.$$

First lemma. *Notations being as above, let L be a finite (not necessarily separable) extension of the field K, and B the integral closure of A in L. Suppose that an element ψ of B have the two following properties:*

(1°) $\psi\equiv 0 \pmod{\pi}$,

(2°) *If for α and β of B we have $\alpha\beta\psi\equiv 0 \pmod{\pi}$, then we have $\alpha\psi\equiv 0$ $\pmod{\pi}$ or $\beta\psi\equiv 0 \pmod{\pi}$.*

Then there exists an extension of V of v to L such that $V(\pi)=V(\psi)+1$.

Here, and in what follows, $\alpha\equiv 0 \pmod{\pi}$ for $\alpha\in B$ means $\alpha\equiv 0 \pmod{\pi B}$, namely, $\alpha/\pi\in B$.

For the proof, we follow Edwards [3], which is extracted from Dedekind [2]. But we must treat also the case where the field L is inseparable over K. First we show that the ring B is completely integrally closed. Suppose that ξ be a non-zero element of B and α an element of L such that $\xi\alpha^n\in B$ for any non-negative integer n. There exists a positive integer q such that the set $L^q=\{x^q\,;\,x\in L\}$ is contained in the separable closure L_s of K in L. Then $\xi^q\alpha^{qn}\in B_s=L_s\cap B$ for any non-negative n. The ring B_s is Noetherian, since the ring A is Noetherian and B_s is the integral closure of A in the separable extension L_s over K. Therefore B_s is completely integrally closed, since it is an integrally closed Noetherian ring. Hence $\alpha^q\in\beta_s\subseteq B$. This implies $\alpha\in B$, since B is integrally closed. Thus, as $\psi/\pi\notin B$, we have shown that for any non-zero element ξ of B there exists