61. On the Numerically Fixed Parts of Line Bundles

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The purpose of this paper is to study the base loci of line bundles. Details will appear elsewhere.

By V we denote a non-singular projective variety defined over an algebraically closed field k. For a line bundle L on V, we have the base locus $\mathrm{Bs}\,|L|$ of the complete linear system and the stable base locus $\mathrm{SBs}\,(L) = \bigcap_{m=1}^{\infty} \mathrm{Bs}\,|mL|$ (Fujita [1]). In this paper, by $\kappa_{num}(L, V) \geqq 0$, we mean that there exist a birational morphism $f: W \to V$, a positive integer m and a nef line bundle S on W such that $H^0(W, mf^*L - S) \neq 0$.

§0. Pseudo-effectivity. Let K stand for a field $\boldsymbol{Q}$ or $\boldsymbol{R}$. A K-1-*cycle* on V is an element of $Z_1(V) \otimes_{\boldsymbol{Z}} K$, where $Z_1(V)$ is a free abelian group generated by irreducible curves on V. A K-1-cycle C is said to be *nef* if $(D, C) \geqq 0$ for any irreducible divisor D on V. A K-line bundle L is said to be *pseudo-effective* if $(L, C) \geqq 0$ for any K-1-cycle C on V.

Proposition 0. *For any $\boldsymbol{Q}$-line bundle L on V, the following conditions are equivalent to each other*:

(1) *L is pseudo-effective.*

(2) *For any ample line bundle A on V, and for any integer $n \geqq 1$, we have $\kappa(A + nL, V) \geqq 0$.*

§1. The numerical base locus of L. We shall introduce the set $\mathrm{NBs}\,(L)$, which may be a numerical analog of $\mathrm{SBs}\,(L)$.

Proposition 1. *Let L be a $\boldsymbol{Q}$-line bundle and let A an ample $\boldsymbol{Q}$-line bundle. Then*

(1) $\mathrm{SBs}\,(A + nL) \subset \mathrm{SBs}\,(A + (n+1)L)$.

(2) $\bigcup_{n=1}^{\infty} \mathrm{SBs}\,(A + nL)$ *does not depend on the choice of A, depending only on L.*

Proof. (1) We take a sufficiently large m. Then mA is very ample and

$$\begin{aligned}\mathrm{SBs}\,(A + nL) &= \mathrm{Bs}\,|m(n-1)(A + nL)| \supset \mathrm{Bs}\,|mA + m(n-1)(A + nL)| \\ &= \mathrm{Bs}\,|nm(A + (n-1)L)| = \mathrm{SBs}\,(A + (n-1)L).\end{aligned}$$

(2) Given two ample $\boldsymbol{Q}$-line bundles A_1 and A_2, we choose $p \gg 0$ such that $pA_2 - A_1$ is very ample. For any $n \geqq 1$ and a sufficiently large $m \geqq 1$, we have

$$\begin{aligned}\mathrm{SBs}\,(A_1 + pnL) &= \mathrm{Bs}\,|m(A_1 + pnL)| \supset \mathrm{Bs}\,|m(pA_2 - A_1) + m(A_1 + pnL)| \\ &= \mathrm{Bs}\,|mp(A_2 + nL)| = \mathrm{SBs}\,(A_2 + nL).\end{aligned}$$

By this,