60. On the Diffeomorphism Types of Elliptic Surfaces

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An *elliptic surface* [3] is a complex surface M with a holomorphic map π of M onto a Riemann surface S such that the inverse image $\pi^{-1}(p)$ of any general point p is an elliptic curve. Matsumoto [4] [5] proved that the diffeomorphism type of M is completely determined by its euler number $e(M)$ and the genus of S if M contains no multiple fiber. (The case when the genus of S is 0 was proved by Kas [2] and Moishezon [6].) The case when M has multiple fibers is more difficult and actually there are examples with exotic smooth structures (Dolgacev surfaces) as was proved by Donaldson [1], Morgan, and Friedman. However we can show that in many cases the diffeomorphism types of the elliptic surfaces are completely determined by their euler numbers and their fundamental groups. By Moishezon [6] we may assume that every singular fiber of the elliptic surfaces with which we are concerned is either a multiple torus ${}_mI_0$ or a fiber of type I_1 ([3]). Let $\pi: M\to S$ be such an elliptic surface. We can consider S as a 2-*orbifold* such that every point p_i which is the image by π of a multiple torus of multiplicity m_i is a cone point of cone angle $2\pi/m_i$ $(i=1, \cdots, k)$. Then we have:

Theorem. *Let π: $M\to S$ and π': $M'\to S'$ be the relatively minimal elliptic surfaces. Suppose that S and S' are either euclidean or hyperbolic. Then M is diffeomorphic to M' if and only* if $e(M)=e(M')$ *and* $\pi_1 M\cong\pi_1 M'$.

This theorem is divided into the following two cases.

Case 1. $e(M)$ $(e(M'))>0$. This implies that $M(M')$ contains at least one singular fiber other than a multiple torus. In this case Theorem also holds when $S(S')$ is spherical with 3 cone points and is derived from:

Claim A. *If S is isomorphic to S' as 2-orbifolds, then M is diffeomorphic to M' if and only if $e(M)=e(M')$.*

Claim B. *If S is not isomorphic to S', then $\pi_1 M\neq\pi_1 M'$.*

Case 2. $e(M)=e(M')=0$. In this case every singular fiber of $M(M')$ is a multiple torus. M and M' are considered as 4-dimensional Seifert fiberings studied by Thornton [8] and Zieshang [9]. Theorem in this case was proved by Zieshang [9] if S and S' are hyperbolic, and was proved by Sakamoto-Fukuhara [7] if $S=S'=T^2$. In the other cases we can see that $\pi_1 M\cong\pi_1 M'$ implies that there is a diffeomorphism between M and M' (not necessarily fiber-preserving). We can also see that there are seven examples each of which admits both the structure of a T^2-bundle over T^2 and