59. On Whittaker Vectors for Generalized Gelfand-Graev Representations of Semisimple Lie Groups

By Hiroshi YAMASHITA
Department of Mathematics, Kyoto University

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Let G be a reductive algebraic group over a local (or a finite) field and $\mathfrak{g}$ its Lie algebra. A regular nilpotent element of $\mathfrak{g}$ gives canonically a non-degenerate character of a maximal unipotent subgroup. The representation of G induced from such a character is called a *Gelfand-Graev representation*, and it is multiplicity free if G is quasi-split. N. Kawanaka [3] generalized this construction using Dynkin's theory on nilpotent Ad(G)-orbits, and associated to every nilpotent orbit an induced representation called *generalized* Gelfand-Graev representation (GGGR). In [3], the GGGRs of finite reductive groups were studied in detail.

1. Definition of GGGRs. Let $G=KAN$ be an Iwasawa decomposition of a connected semisimple Lie group G with finite center, and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$ the corresponding decomposition of its Lie algebra $\mathfrak{g}$. Denote by W the Weyl group of $(\mathfrak{g}, \mathfrak{a})$. Choose a positive system Λ^+ of the root system Λ of $(\mathfrak{g}, \mathfrak{a})$ so that $\mathfrak{n}=\sum_{\lambda\in\Lambda^+}\mathfrak{g}_\lambda$, where $\mathfrak{g}_\lambda$ denotes the root space of λ. Let U be the maximal unipotent subgroup with Lie algebra $\mathfrak{u}=\sum_{\lambda\in\Lambda^+}\mathfrak{g}_{-\lambda}$.

For a C^∞-manifold Ω and a Fréchet space E, let $C^\infty(\Omega, E)$ (resp. $C_0^\infty(\Omega, E)$) denote the space of E-valued smooth functions on Ω (resp. those with compact supports) equipped with the Schwartz topology. Let V be a closed subgroup of G and η a smooth representation (see e.g. [4, p. 254]) of V on a Fréchet space E. The left translation defines a smooth representation π_η of G on the space $C_\eta^\infty(G, E)$ of f in $C^\infty(G, E)$ satisfying $f(gv)=\eta(v)^{-1}f(g)$ $(g\in G, v\in V)$, which is equipped with the topology inherited from that of $C^\infty(G, E)$.

For a non-zero nilpotent element $X\in\mathfrak{g}$, by Jacobson-Morozov theorem, there exists an $\mathfrak{sl}_2$-triplet $\{X, H, Y\}\subset\mathfrak{g}$ containing X: $[H, X]=2X$, $[H, Y]=-2Y$, $[X, Y]=H$. By taking a suitable Ad(G)-conjugate of X, we may assume that $-H$ is dominant in $\mathfrak{a}$. Since $-\lambda(H)=0$, 1 or 2 for any simple root λ, we get a gradation $\mathfrak{g}=\sum_{i\in Z}\mathfrak{g}(i)$ by ad(H). For $i\geqq 1$, $\mathfrak{u}(i)=\sum_{k\geqq i}\mathfrak{g}(k)$ is a Lie subalgebra of $\mathfrak{u}$. Since $\mathfrak{g}(i)$ and $\mathfrak{g}(j)$ are orthogonal with respect to the Killing form B of $\mathfrak{g}$ if $i+j\neq 0$, there exists a subalgebra $\mathfrak{u}(1.5)$ of $\mathfrak{u}(1)$ which has following two properties: (i) $\mathfrak{u}(2)\subseteqq\mathfrak{u}(1.5)$ and $2\dim\mathfrak{u}(1.5)=\dim\mathfrak{u}(1)+\dim\mathfrak{u}(2)$, (ii) $B(Y, [\mathfrak{u}(1.5), \mathfrak{u}(1.5)])=(0)$. Then we can define a unitary character η_X of $U(1.5)=\exp\mathfrak{u}(1.5)$ by $\eta_X(\exp Z)=\exp\sqrt{-1}B(Y, Z)$ for $Z\in\mathfrak{u}(1.5)$.

Definition. For a non-zero nilpotent element $X\in\mathfrak{g}$, the smooth repre-