58. *Local Isometric Embedding of Two Dimensional Riemannian Manifolds into R^3 with Nonpositive Gaussian Curvature*

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As it is well known, the problem of C^∞ local isometric embedding of a two dimensional Riemannian manifold into $\boldsymbol{R}^3$ is a problem whether C^∞ functions $x(u,v)$, $y(u,v)$, $z(u,v)$ which satisfy

(1) $$dx^2+dy^2+dz^2=Edu^2+2Fdudv+Gdv^2$$

exist in a neighborhood of a point, say $(u,v)=0$, when the first fundamental form $Edu^2+2Fdudv+Gdv^2$ is given. The results already known are as follows. Let K be the Gaussian curvature of the two dimensional manifold, then the classical result is that the problem is affirmatively answered if $K\neq 0$ at $(u,v)=0$, and a recent interesting result due to Lin [3] is that it is also affirmative if $K=0$, $grad\,K\neq 0$ at $(u,v)=0$. Now a natural question arises. Namely, is it affirmative when $K=grad\,K=0$ at $(u,v)=0$ and one of the following conditions holds:

(i) $Hess\,K(0,0)>0$,

(ii) $Hess\,K(0,0)<0$,

(iii) $Hess\,K(0,0)$ has two eigenvalues with opposite signs?

Hereafter, for simplicity, we refer to the case with conditions $K=grad\,K=0$ at $(u,v)=0$ and (i) (resp. (ii) and resp. (iii)) by (i) (resp. (ii) and resp. (iii)).

Then what we have obtained is the following.

Theorem. *The problem of C^∞ local isometric embedding is also affirmative in the case* (ii).

The idea of the proof is as follows. Since a two dimensional Riemannian manifold whose Gaussian curvature is zero is locally isometric to Euclidean space with its standard metric, it is enough to solve the following equation (2) for z under the condition $\nabla z(0,0)=0$, which assures the Gaussian curvature of the metric

$$Edu^2+2Fdudv+Gdv-dz^2$$

vanishes in a neighborhood of $(u,v)=0$. Namely,

(2) $$(z_{11}-\Gamma^i_{11}z_i)(z_{22}-\Gamma^i_{22}z_i)-(z_{12}-\Gamma^i_{12}-z_i)^2 =K(EG-F^2-Ez_2^2-Gz_1^2+2Fz_1z_2)$$

where Γ^i_{jk} are Christoffel symbols, z_i is the first derivative of z with respect to the i-th variable and z_{jk} is the second derivative with respect to the j-th and k-th variables by calling u the first variable and v the second variable.

Now we construct an approximate solution $\bar{z}$ which satisfies (2) modulo a certain term with flatness $O(u^4)$ and linearize (2) at $\bar{z}$. Then the linearized