56. A Characterization of Almost Automorphic Functions

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Recently R. A. Johnson gave us a linear almost periodic differential equation with an almost automorphic solution which is not almost periodic [1]. In this paper we study almost automorphic functions and obtain a characterization of them by using Veech's result and Levitan's N-almost periodic functions.

We denote the set of real numbers by R. Let X be a metric space with the metric d_X. A continuous mapping $\pi: X\times R\to X$ is called a *flow on (a phase space) X* if π satisfies following two conditions:

(1) $\pi(x, 0)=x$ for $x\in X$.

(2) $\pi(\pi(x, t), s)=\pi(x, t+s)$ for $x\in X$ and $t, s\in R$.

The orbit through $x\in X$ of π is denoted by $C_\pi(x)$. $M\subset X$ is called an invariant set of π if $C_\pi(x)\subset M$ for every $x\in M$. The restriction of π to an invariant set M of π is denoted by $\pi|M$. A non-empty compact invariant set M of π is called a *minimal set of π* if $\overline{C_\pi(x)}=M$ for every $x\in M$, where $\overline{C_\pi(x)}$ is the closure of $C_\pi(x)$. If X is itself a minimal set, we say that π is a *minimal flow on X*. A flow π is said to be *equicontinuous* if for each $\varepsilon>0$ there exists a $\delta>0$ such that $d_X(\pi(x, t), \pi(y, t))<\varepsilon$ for $x, y\in X$ with $d_X(x, y)<\delta$ and for $t\in R$. A point $x\in X$ is called an *almost automorphic point* if for each sequence $\{t_n\}\subset R$ there exists a subsequence $\{t_{n_k}\}\subset\{t_n\}$ such that $\pi(x, t_{n_k})\to y\in X$ and $\pi(y, -t_{n_k})\to x$ as $k\to\infty$. We denote the set of almost automorphic points of π by $A(\pi)$. We can easily see that if $x\in A(\pi)$, then $\overline{C_\pi(x)}$ is a minimal set of π, and that $A(\pi)$ is an invariant set of π. A minimal flow π is said to be *almost automorphic* if $A(\pi)\neq\phi$. Let π be a minimal flow on X. $\lambda\in R$ is called an *eigenvalue of π* if there exists a continuous function $\chi: X\to K$ such that the relation $\chi(\pi(x, t))=\chi(x)\exp(2\pi i\lambda t)$ holds for $(x, t)\in X\times R$, where K is the unit circle in the complex plane. In this case χ is called an *eigenfunction of π belonging to λ*. We denote the set of eigenvalues of π by $\Lambda(\pi)$. It is well known that $\Lambda(\pi)$ is a countable subgroup of R for any minimal flow.

Proposition 1. *Let π be an equicontinuous minimal flow on X. Then, if a sequence $\{t_n\}\subset R$ satisfies that $\lim_{n\to\infty}\exp(2\pi i\lambda t_n)=1$ for every $\lambda\in\Lambda(\pi)$, then we have $\pi(x, t_n)\to x$ as $n\to\infty$ for $x\in X$.*

Proof. We denote the eigenfunction of π belonging to $\lambda\in\Lambda(\pi)$ by χ_λ. Since π is equicontinuous, it is well known that, if $\chi_\lambda(x)=\chi_\lambda(y)$ $(x, y\in X)$ for every $\lambda\in\Lambda(\pi)$, then we have $x=y$. Let $x\in X$. We assume that