47. The Existence of Spectral Decompositions in L^p-Subspaces

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1. Introduction. In this note we outline the main results of a forthcoming paper [4]. Throughout we suppose that μ is an arbitrary measure, $1<p<\infty$, and Y is a subspace of $L^p(\mu)$. An invertible operator $V\in\mathcal{B}(Y)$ will be called *power-bounded* provided $\sup_{n\in \boldsymbol{Z}}\|V^n\|<\infty$, where $\boldsymbol{Z}$ denotes the additive group of integers. We show that $\{V^n\}_{n=-\infty}^{\infty}$ is automatically the Fourier-Stieltjes transform of a spectral family of projections concentrated on $[0, 2\pi]$ (see [1, § 2] for definitions and the Riemann-Stieltjes integration theory of spectral families). We deduce that every bounded, one-parameter group on Y is the Fourier-Stieltjes transform of a spectral family of projections $E(\cdot): \boldsymbol{R}\to\mathcal{B}(X)$. This result generalizes work in [2], [8], and can be used to obtain a complete analogue for $L^p(\mathcal{K})$ of Helson's correspondence [10, § 2.3] between cocycles and the normalized, simply invariant subspaces of $L^2(\mathcal{K})$, where $\mathcal{K}$ is a compact abelian group with archimedean ordered dual. In particular, in $L^p(\mathcal{K})$ every such invariant subspace is the range of a bounded projection.

2. Abstract results. An operator U on a Banach space X is called *trigonometrically well-bounded* [3] provided

$$U=\int_{[0,2\pi]}^{\oplus} e^{i\lambda}dE(\lambda)$$

for a spectral family of projections $E(\cdot): \boldsymbol{R}\to\mathcal{B}(X)$ such that the strong left-hand limits $E(0^-)$, $E((2\pi)^-)$ are 0, I, respectively. $E(\cdot)$ is necessarily unique, and will be called the *spectral decomposition* of U. Let $BV(\boldsymbol{T})$ be the Banach algebra of complex-valued functions having bounded variation on the unit circle. For $f\in BV(\boldsymbol{T})$ put

$$F_1(t)=\lim_{s\to t+} f(e^{is}),\qquad F_2(t)=\lim_{s\to t-} f(e^{is})$$

for $t\in\boldsymbol{R}$, and let $\hat{f}$ be the Fourier transform of f.

(2.1) **Theorem.** *Let $U\in\mathcal{B}(X)$ be trigonometrically well-bounded and power-bounded, and suppose $f\in BV(\boldsymbol{T})$. Then $\sum_{n=-N}^{N}\hat{f}(n)U^n$ converges in the strong operator topology, as $N\to+\infty$, to*

$$2^{-1}\int_{[0,2\pi]}^{\oplus}(F_1+F_2)dE,$$

where $E(\cdot)$ is the spectral decomposition of U.

Proof. For $t\in\boldsymbol{R}$, $x\in X$, let

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