28. On Semi-idempotents in Group Rings

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After Gray [1], an element $\alpha \neq 0$ of a ring R is called *semi-idempotent* if and only if α is not in the proper two-sided ideal of R generated by $\alpha^2-\alpha$, i.e. $\alpha \notin R(\alpha^2-\alpha)R$ or $R=R(\alpha^2-\alpha)R$. 0 is also counted among semi-idempotents. It is obvious that idempotent element is semi-idempotent. Throughout this note, K denotes a (commutative) field. We are concerned here with the group ring $R=KG$ over a group G. § 1 contains some propositions of general nature. In § 2 we prove a theorem for the case where G is abelian.

§ 1. Trivial and non-trivial semi-idempotents. In the following, we consider the group ring $R=KG$, $G\neq 1$. It is easily seen that for $k \in K$ the element $k\cdot 1 \in R$ is semi-idempotent. Semi-idempotents of this form are called *trivial*, other semi-idempotents *non-trivial*. The subset $\{\sum_{g \in G} a_g g;\ \sum_{g\in G} a_g=0\}$ forms a proper two-sided ideal of R, called the *augmentation ideal* $w(R)$ of R (Passman [2]).

Proposition 1. *The group ring $R=KG$ $(G\neq 1)$ contains non-trivial semi-idempotents.*

Proof. Any element g of $G-\{1\}$ is non-trivial semi-idempotent because $g \notin w(R)$, $g^2-g \in w(R)$.

Proposition 2. *If H is a subgroup of G of finite order n, $\alpha=(\sum_{h\in H} h)+1$ is a non-trivial semi-idempotent.*

Proof. We have $\alpha^2-\alpha=(n+1)\sum_{h\in H} h$. If $n+1=0$ in K, α is idempotent. If $n+1\neq 0$ in K, we have $R(\alpha^2-\alpha)R=R(\sum_{h\in H} h)R$, so that $\alpha \in R(\alpha^2-\alpha)R$ implies $1=\alpha-\sum_{h\in H} h \in R(\alpha^2-\alpha)R$ whence $R=R(\alpha^2-\alpha)R$. Thus α is semi-idempotent.

Proposition 3. *If α is non-trivial idempotent of $R=KG$ (i.e. $\alpha \in R$, $\alpha^2=\alpha$ and $\alpha \notin \{0, 1\}$), $\alpha+1$ is semi-idempotent.*

Proof. Put $\beta=\alpha+1$. Then we have $\beta^2-\beta=\alpha\beta=\alpha^2+\alpha=2\alpha$. If $2=0$ in K, β is idempotent. If $2\neq 0$ in K, we have $R(\beta^2-\beta)R=R\alpha R$. Therefore $\beta \in R(\beta^2-\beta)R$ implies $\alpha+1 \in R\alpha R$, $R(\beta^2-\beta)R=R$. Thus β is semi-idempotent.

§ 2. Abelian case. Now we consider the case where $R=KG$ is a group ring over an abelian group G. Then every ideal in R is of course two-sided.

Proposition 4. *Let $R=KG$ be a group ring over an abelian group G. If α $(\neq 0)$ is semi-idempotent but not a unit in R, then $\alpha-1$ is not a unit in R.*

Proof. Suppose $\alpha-1$ be a unit in R. Then there is an element β of