69. On the Ito Formula of Noncausal Type

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Let $\{B(x,w); x\geqq 0\}$ be the real Brownian motion defined on a probability space $(W,\mathcal{F},P)$ and let $\{\phi_n\}$ be an orthonormal basis in the real Hilbert space $L^2(0,1)$. Following the article [1], we say that a real random function $f(x,w)$, satisfying the condition

$$P\left[\int_0^1 f^2(x,w)dx<\infty\right]=1,$$

is integrable with respect to the basis $\{\phi_n\}$ on a measurable set $A\subset[0,1]$, if the series

$$\sum_n \int_A f(x,w)\phi_n(x)\,dx\int_0^1 \phi_n(x)dB(x)$$

converges in probability. In this case, we shall denote the sum by

$$\int_A f\, d_\phi B(x)$$

and call such integral the stochastic integral of noncausal type.

Since this integral can also apply to those random functions which are not adapted to the family of σ-fields, $\mathcal{F}_x=\sigma(B(y,w); y\leqq x)$ $(x\geqq 0)$, it is meaningful to consider the stochastic integral equation of noncausal type:

$$(1)\qquad X(x,w)-\xi(w)=\int_0^x a(y,X(y,w))dy+\int_0^x b(y,X(y,w))d_\phi B(y),$$

where $\xi(w)$ is a real random variable and $a(x,y)$, $b(x,y)$ $((x,y)\in[0,1]\times R^1)$ are some functions. As for the equation (1), Ogawa [2] has shown the existence of solutions by constructing one for a specified basis (see Theorem below). Our aim in this paper is to show that the constructed solution satisfies a formula of Ito's type in the noncausal case.

We begin by summarizing his result. Assume that the functions $a(x,y)$ and $b(x,y)$ satisfy the following two conditions:

(H, 1) The function $a(x,y)$ belongs to the class C^1 and $b(x,y)$ to the class C^2. Moreover, $b(x,y)$ is thrice continuously differentiable in y.

(H, 2) For each real number r the stochastic integral equation:

$$(2)\qquad Y(x,w)-r=\int_0^x a(y,Y(y,w))dy+\int_0^x b(y,Y(y,w))dB(y),$$

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