68. On the Banach-Saks Property

By Nolio OKADA
Department of Mathematics, Science University of Tokyo

(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 12, 1984)

1. Introduction. According to Banach and Saks [1], every bounded sequence in $L^p(0,1)$ or l^p $(1<p<\infty)$ has a subsequence whose Cesàro-means converge strongly. More generally every uniformly convex Banach space possesses this so-called *Banach-Saks property*, as shown by Kakutani [4]. In particular every Hilbert space has this property. In nonlinear analysis, by utilizing a duality mapping some assertions which are valid in the case of Hilbert spaces are extended to the case of special classes of Banach spaces. Especially in the case of Banach spaces with a uniformly convex conjugate space, such extentions are often obtained since a duality mapping is uniformly strongly continuous on each bounded subset of such a Banach space (see Browder [2, p. 42] or Kato [5]). So we consider whether such a Banach space has the Banach-Saks property or not. The result is positively extended and is stated as follows:

Theorem. *Let X be a Banach space with a uniformly convex conjugate space X^*. Then X possesses the Banach-Saks property.*

After we have proved the above theorem, we find the following result due to Enflo [3]:

For a Banach space X with a conjugate space X^*, X is uniformly convexifiable if and only if X^* is uniformly convexifiable.

Hence, combining this result and Kakutani's theorem we can get our theorem. However, our method of the proof is based on very elementary facts about a duality mapping and there might be some interest in the simplicity of the construction of a subsequence whose Cesàro-means converge strongly.

The author wishes to express his sincere gratitude to Prof. T. Shibata and Prof. S. Miyajima for their advice and encouragement.

2. Proof of the theorem.

Proof of the theorem. Since X^* is uniformly convex, for each x in X there exists a unique $F(x)$ in X^* such that

$$(x, F(x)) = \|x\|\,\|F(x)\| \quad \text{and} \quad \|x\| = \|F(x)\|,$$

where $(\cdot,\cdot)$ denotes the canonical pairing of X and X^*. The mapping $F: X \to X^*$ is the so-called duality mapping. As we stated in the introduction, the uniform convexity of X^* also implies that F is uniformly strongly continuous on each bounded subset of X. Since X