75. On Certain Cubic Fields. III

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1. The notations E_F, E_F^+, $\mathcal{O}_F$ for an algebraic number field F, D_g for a polynomial $g(x)\in \boldsymbol{Z}[x]$ and $D(\theta)$ for an algebraic number θ have the same meanings as in [1]. For a totally real cubic field K, we also use the notations $\mathcal{A}(K)$, $\mathcal{B}_\varepsilon(K)$ and $S: K\to \boldsymbol{R}$ as in [1].

The purpose of this note is to show the following theorem:

Theorem. *Let* $K=\boldsymbol{Q}(\delta)$, *where* $\mathrm{Irr}(\delta:\boldsymbol{Q})=g(x)=x^3-nx^2-(n+1)x-1$, $n\in \boldsymbol{Z}$ *but* $n\neq 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, -6$. *If* $D_g=(n^2+n-3)^2-32$ *is square free, then we have* $\delta\in\mathcal{A}(K)$, $\delta+1\in\mathcal{B}_\delta(K)$ *and* $E_K^+=\langle\delta, \delta+1\rangle$.

Remark 1. We may limit our consideration to the case $n\leqq -7$ for the following reason. Put $G(n, x)=x^3-nx^2-(n+1)x-1$ and $m=-(n+1)$. Then we have $-(1/x^3)G(n, x)=G(m, 1/x)$ and if $n\geqq 6$, we have $m\leqq -7$. Thus if $\mathrm{Irr}(\delta:\boldsymbol{Q})=G(n, x)$ with $n\geqq 6$, then $\mathrm{Irr}(1/\delta:\boldsymbol{Q})=G(m, x)$ with $m\leqq -7$. Thus we suppose $n\leqq -7$ in the sequel.

Remark 2. $K/\boldsymbol{Q}$ is cubic because of the irreducibility of $g(x)$, and it is totally real in virtue of $D_g=(n^2+n-3)^2-32>0$. It is easy to verify that $(n^2+n-3)^2-32$ can not be a square. Thus $K/\boldsymbol{Q}$ is non Galois.

2. *Proof of Theorem.* First we shall show $\delta\in\mathcal{A}(K)$, $\delta+1\in\mathcal{B}_\delta(K)$. It is clear that $\delta, \delta+1\in E_K^+$. As $K=\boldsymbol{Q}(\delta)$, $D_g\neq 0$ and D_g is square free, we have $D_g=D(\delta)$ and consequently we have $\mathcal{O}_K=\boldsymbol{Z}+\boldsymbol{Z}\delta+\boldsymbol{Z}\delta^2$. Any unit $v\neq 1$ in E_K^+ can be written as $v=a+b\delta+c\delta^2$, where $a, b, c\in\boldsymbol{Z}$ and $(b, c)\neq(0, 0)$. This yields, in denoting the conjugates of δ by α, β, γ,

$$\begin{aligned}S(v)=\frac{1}{2}\{&b^2(\alpha-\beta)^2+c^2(\alpha^2-\beta^2)^2+2bc(\alpha-\beta)(\alpha^2-\beta^2)\\&+b^2(\beta-\gamma)^2+c^2(\beta^2-\gamma^2)^2+2bc(\beta-\gamma)(\beta^2-\gamma^2)\\&+b^2(\gamma-\alpha)^2+c^2(\gamma^2-\alpha^2)^2+2bc(\gamma-\alpha)(\gamma^2-\alpha^2)\}.\end{aligned}$$

Using Proposition 4 in [1], we have $S(\delta)=n^2+3n+3>0$ and $S(v)=P+Q+R$, where

$$P=\frac{1}{2}b^2\{(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2\}=b^2S(\delta),$$

$$\begin{aligned}Q&=\frac{1}{2}c^2\{(\alpha^2-\beta^2)^2+(\beta^2-\gamma^2)^2+(\gamma^2-\alpha^2)^2\}=c^2(n^4+4n^3+5n^2+8n+1)\\&=c^2S(\delta)+(n^4+4n^3+5n-2)c^2=(n^2+n+1)c^2S(\delta)+(-2n^2+2n-2)c^2,\end{aligned}$$