68. A Note on Circumferentially Mean Univalent Functions in an Annulus

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1. Introduction. In the previous paper [1] we extended the so-called Montel-Bieberbach's theorem on values omitted by meromorphic and univalent functions in $|z|<1$, to the case of circumferentially mean univalence (defined hereafter). In the next paper [2] we announced the results on meromorphic and circumferentially mean univalent functions in an annulus which mean an extension of the author's results [1]. In this paper, we shall first extend Grötzsch's theorem ([3] or [5]) to the case of circumferentially mean univalence and then prove the author's results [2] in the precise and intrinsic form.

We shall define circumferentially mean univalent functions in a domain D. Let $f(z)$ be regular or meromorphic in D and $n(R, \Phi)$ denote the number of roots of the equation $f(z)=w=Re^{i\Phi}$. We define $p(R)$ as follows.

$$p(R)=\frac{1}{2\pi}\int_0^{2\pi} n(R, \Phi)d\Phi \quad (0\leq R<\infty).$$

If $p(R)\leq 1$ $(0\leq R<\infty)$, $f(z)$ is called "circumferentially mean univalent".

2. We shall first state the following two lemmas.

Lemma 1. *Let $w=f(z)$ be single-valued, regular in $1\leq|z|<R$ and $|f(z)|\leq 1$ there. Moreover let the circle $|z|=1$ be univalently mapped onto the circle $|w|=1$. If we denote the harmonic measure of the circle $|z|=1$ with respect to the annulus $1<|z|<R$ by $\omega(z)$ and do the harmonic measure of $|w|=1$ with respect to the image domain D_f under $w=f(z)$ by $\omega_f(w)$, then we have*

$$(1) \qquad I(\omega(z))\geq I(\omega_f(w)),$$

where $I(\omega(z))$ or $I(\omega_f(w))$ denote the Dirichlet integral of $\omega(z)$ or $\omega_f(w)$ respectively.

Proof. We may consider Landau-Osserman's results [6] by means of exhaustion method.

Lemma 2. *Let $f(z)$ satisfy the same conditions as in Lemma 1 and D_f, or $\omega_f(w)$ denote the same notation in Lemma 1 respectively. If D_f^* denotes the circularly symmetrized domain of D_f with respect to*