43. Pluricanonical Mappings of Canonically Polarized Varieties

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In this note, we shall prove the following result.

Theorem. *Let V be a canonically polarized variety of dimension n over $\mathbf{C}$. Then there exists an integer N which depends only on n such that the m-th canonical mappings Φ_m of V are birational for all $m \geqq N$.*

Here, V is said to be canonically polarized, if it is non-singular, complete and if the canonical divisor $K(V)$ is ample.

To prove this, we need the following lemmas.

Lemma 1 (Matsusaka). *Let V be a canonically polarized variety of dimension n and let γ be $K(V)^{(n)}$. If $P_m(V) \geqq \gamma m^{n-1}+n$, then the m-th canonical mapping Φ_m is generically finite.*

Lemma 2 (Wilson). *Let V be a non-singular variety of dimension n. If there exists m such that the m-th canonical mapping Φ_m is generically finite and $P_m(V) \geqq n+2$ then Φ_{nm+1} is birational.*

Lemma 3. *Let V be a complete non-singular variety of dimension n over a field of characteristic zero. Assume that the m_b-th canonical mapping is birational. Then the m-th canonical mapping is birational for all $m \geqq \mathrm{Max}\{1, nm_b(m_b-1)\}$.*

Proof. Put $W_m = \Phi_m(V)$. Clearly, $\mathrm{Rat}(W_{km_b}) = \mathrm{Rat}(W_{m_b}) = \mathrm{Rat}(V)$ for all integers $k \geqq 1$. By Wilson's Lemma $\mathrm{Rat}(W_{nm_b+1}) = \mathrm{Rat}(V)$. It suffices to show that we can find integers $\alpha, \beta \geqq 0$ such that $m = \alpha(nm_b+1)+\beta m_b$. In fact, we can find integers $q \geqq 1$, $nm_b(m_b-1) > r \geqq 0$ such that $m = qnm_b(m_b-1)+r$. Also, $r = sm_b+\alpha$ for $s \geqq 0$, $m_b > \alpha \geqq 0$. Hence $m-\alpha(nm_b+1) = \beta m_b$, where $\beta = n(q(m_b-1)-\alpha)+s$. Note that $\beta \geqq 0$.

Proof of Theorem. Since $K(V)$ is ample, it follows that $P_m(V) = \chi(V, \mathcal{O}(mK)) = \Sigma_{i=0}^{n}(-1)^i \dim H^i(V, \mathcal{O}(mK))$ for $m \geqq 2$. Note that the leading coefficient of polynomial $\chi(V, \mathcal{O}(mK))$ is equal to $\gamma/n!$ Moreover if $P_k(V) > \gamma k^{n-1}+n-1$ (Matsusaka inequality) for one of value k such that $2 \leqq k \leqq n+2$, then we can find such a number N that all the m-th canonical mappings are birational for all $m \geqq N$, by virtue of Lemmas 1, 2 and 3.

Case 1. Assume $\gamma \leqq n-1$. If $P_m(V) > (n-1)(m^{n-1}+1)$ for one value m such that $2 \leqq m \leqq n+2$, then Matsusaka inequality holds. Hence