39. A Proposition on the Cardinality of Closed Discrete Subsets of a Topological Space

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D. B. Gauld and M. K. Vamanamurthy [3] have considered on a problem of the cardinality of a closed discrete subset of a separable normal space. In this paper, we prove that the cardinality of each closed discrete subset of a separable normal space being at most countable is independent of the usual axioms of set theory, i.e. ZFC.

Theorem 1. *If X is a normal space, then the cardinality of each closed discrete subspace of X is less than the exponential of its density.*

Proof. Let D be a dense subset of X having the cardinality of the density of X. Assume C is a closed discrete subspace of X, moreover A is a subset of C. Since C is closed discrete, both A and $C-A$ are closed subsets of X. Then by the normality of X, there exist disjoint open sets U_A and V_A in X such that A is contained in U_A and $C-A$ is contained in V_A. Next define a mapping f from the power set of C to the power set of D such that for each subset A of C, $f(A) = D \cap U_A$. Then clearly f is a one-to-one mapping. Hence $\exp|C| \leqq \exp|D|$. Therefore, $|C| < \exp|D|$. We complete the proof.

Remark. In general, it is not always $\kappa \leqq \lambda$, whenever $\exp\kappa \leqq \exp\lambda$. For example, if Martin's axiom and the negation of the continuum hypothesis are assumed, then $\exp\aleph_0 = \exp\aleph_1$, but $\aleph_0 < \aleph_1$. Hence we can construct a separable normal space having uncountable closed discrete subsets assuming Martin's axiom and the negation of the continuum hypothesis. See Theorem 2.

Corollary 1. *If the generalized continuum hypothesis is assumed, then the cardinality of each closed discrete subset of a normal space is less than or equal to its density.*

Proof. Let κ be the density of a normal space X. Then $\exp\kappa$ equal to κ^+ by the assumption. Hence the cardinality of each closed discrete subset of X is less than or equal to κ by Theorem 1. We complete the proof.

We can prove the next result in a similar way to Corollary 1.

Corollary 2. *If the continuum hypothesis is assumed, then the cardinality of each closed discrete subset of a separable normal space is countable.*

Corollary 3. *The Sorgenfrey's square S is not normal.*