2. An Asymptotic Formula for the Eigenvalues of the Laplacian in a Domain with a Small Hole

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§1. **Introduction.** This note is a continuation of our previous paper [1]. Let Ω be a bounded domain in $\mathbf{R}^3$ with C^∞ boundary γ and w be a fixed point in Ω. For any sufficiently small $\varepsilon>0$, let B_ε be the ball defined by $B_\varepsilon=\{z\in\Omega\,;\,|z-w|<\varepsilon\}$. Let Ω_ε be the bounded domain defined by $\Omega_\varepsilon=\Omega\backslash\bar{B}_\varepsilon$. Then $\partial\Omega_\varepsilon=\gamma\cup\partial B_\varepsilon$.

Let $0>\mu_1(\varepsilon)\geq\mu_2(\varepsilon)\geq\cdots$ be the eigenvalues of the Laplacian in Ω_ε under the Dirichlet condition on $\partial\Omega_\varepsilon$. Let $0>\mu_1\geq\mu_2\geq\cdots$ be the eigenvalues of the Laplacian in Ω under the Dirichlet condition on γ. We arrange them repeatedly according to their multiplicities.

The aim of this note is to give an asymptotic expression of $\mu_j(\varepsilon)$ as ε tends to zero. We need some notations to state the main result.

Let $G(x,y)$ be the Green function of the Laplacian in Ω satisfying

$$\Delta_x G(x,y)=-\delta(x-y)\qquad x,y\in\Omega,$$
$$G(x,y)|_{x\in\gamma}=0\qquad y\in\Omega.$$

Then the Robin constant τ $(=\tau(w))$ at w is defined by

$$\tau=\lim_{x\to w}(G(x,w)-(4\pi)^{-1}|x-w|^{-1}).$$

Let $\mathbf{G}$ be the Green operator defined by

$$(1.1)\qquad (\mathbf{G}f)(x)=\int_\Omega G(x,y)f(y)dy$$

for $x\in\Omega$.

We have the following

Theorem 1. *Fix j. Assume that the multiplicity of μ_j is one, then*

$$(1.2)\qquad \begin{aligned}\mu_j(\varepsilon)-\mu_j=&-(\tau+(4\pi\varepsilon)^{-1})^{-1}\varphi_j(w)^2\\&-(\tau+(4\pi\varepsilon)^{-1})^{-2}e_j(w)\varphi_j(w)+O(\varepsilon^{5/2})\end{aligned}$$

as ε tends to zero. Here $\varphi_j(x)$ denotes the eigenfunction of the Laplacian under the Dirichlet condition on γ satisfying

$$\int_\Omega \varphi_j(x)^2dx=1.$$

And here

$$(1.3)\qquad e_j(w)=\lim_{x\to w}(G(x,w)\varphi_j(w)+\psi(x)),$$

where $\psi\in L^2(\Omega)$ is the unique solution of

$$(1.4)\qquad ((\mathbf{G}+(1/\mu_j))\psi)(x)=-(1/\mu_j)G(x,w)\varphi_j(w)-(1/\mu_j^2)\varphi_j(w)^2\varphi_j(x)$$

and