21. *On the Trotter Product Formula*

By Tetsuya KOYAMA and Takashi ICHINOSE
Department of Mathematics, Hokkaido University

(Communicated by Kôsaku YOSIDA, M. J. A., Feb. 12, 1981)

Introduction. Kato [5] (cf. Kato-Masuda [8]) proved the Trotter product formula $s\text{-}\lim_{n\to\infty}[e^{-tA/n}e^{-tB/n}]^n = e^{-t(A\dot{+}B)}P$ for the form sum $A\dot{+}B$ of self-adjoint operators A and B which are bounded from below in a Hilbert space $\mathcal{H}$. Here P is the orthogonal projection of $\mathcal{H}$ onto the closure of $\mathcal{D}(|A|^{1/2})\cap\mathcal{D}(|B|^{1/2})$. The purpose of this paper is to extend this result to prove a product formula for the form sum of self-adjoint operators which are not necessarily bounded from below. The product formula obtained involves a "truncation" procedure.

1. Notations and results. First we consider the case of two operators. Let A and B be self-adjoint operators in a Hilbert space $\mathcal{H}$ with spectral families $\{E_A(\lambda)\}$ and $\{E_B(\lambda)\}$, respectively. Let A_+ and A_- be the positive and negative parts of A, i.e. $A_+ = AE_A([0, \infty)) \geqslant 0$, $A_- = -AE_A((-\infty, 0)) \geqslant 0$, and $A = A_+ - A_-$. Define B_+ and B_- similarly for B.

Assume that $\mathcal{D}(A_+^{1/2})\subset\mathcal{D}(B_-^{1/2})$ and $\mathcal{D}(B_+^{1/2})\subset\mathcal{D}(A_-^{1/2})$, and that there exist constants $\alpha\geqslant 0$ and $0\leqslant\beta<1$ such that

$$\begin{aligned} &\|A_-^{1/2}u\|^2 \leqslant \alpha\|u\|^2 + \beta\|B_+^{1/2}u\|^2, \qquad u\in\mathcal{D}(B_+^{1/2}),\\ &\|B_-^{1/2}u\|^2 \leqslant \alpha\|u\|^2 + \beta\|A_+^{1/2}u\|^2, \qquad u\in\mathcal{D}(A_+^{1/2}). \end{aligned} \tag{1}$$

Set $\mathcal{D} = \mathcal{D}(A_+^{1/2})\cap\mathcal{D}(B_+^{1/2})$, and let P be the orthogonal projection of $\mathcal{H}$ onto the closure $\overline{\mathcal{D}}$ of $\mathcal{D}$. Then the quadratic form

$$u \mapsto \|A_+^{1/2}u\|^2 + \|B_+^{1/2}u\|^2 - \|A_-^{1/2}u\|^2 - \|B_-^{1/2}u\|^2, \qquad u\in\mathcal{D}, \tag{2}$$

is bounded from below and closed. The *form sum* of A and B is defined as the self-adjoint operator in the Hilbert space $\overline{\mathcal{D}}$ associated with (2) and denoted by $A\dot{+}B$.

For each $0<\tau\leqslant\infty$, $\mathcal{F}(\tau)$ is the class of bounded real-valued functions $h(t, \lambda)$ on $[0, \tau)\times\boldsymbol{R}$ satisfying the following conditions:

(i) for each fixed λ, $h(t, \lambda)$ is continuous in t at $t=0$ with
$$h(0, \lambda)=1, \qquad (\partial/\partial t)h(0, \lambda) = -\lambda;$$

(ii) for each fixed t, $h(t, \lambda)$ is Borel measurable in λ with
$1\leqslant h(t, \lambda)$ for $\lambda<0$, $h(t, 0)=1$ and $0\leqslant h(t, \lambda)\leqslant 1$ for $\lambda>0$;

(iii) there is a constant M such that $|1-h(t, \lambda)|\leqslant Mt|\lambda|$, $0\leqslant t<\tau$, $\lambda\in\boldsymbol{R}$.

The main result is the following product formula.

Theorem 1. *Let $f(t, \lambda)$ and $g(t, \lambda)$ be in $\mathcal{F}(\tau)$ for some $0<\tau\leqslant\infty$, and assume that there exists a constant $z>1$ such that*