8. On Conformal Diffeomorphisms between Product Riemannian Manifolds

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In this note, a conformal diffeomorphism means a non-homothetic conformal one. Let $M=M_1\times M_2$ and $M^*=M_1^*\times M_2^*$ be connected product Riemannian manifolds of dimension $n\geqq 3$, and denote the metric product structures by (M, g, F) and (M^*, g^*, G) respectively. Several geometers [1]–[3], [5]–[8] proved non-existence of global conformal diffeomorphism between complete product Riemannian manifolds with certain properties. The purpose of this note is to announce the following

Theorem. *If M and M^* are complete product Riemannian manifolds, then there is no global conformal diffeomorphism of M onto M^* such that it does not commute the product structures F and G, $FG\neq GF$, somewhere in M.*

This is an improvement of the main theorem in a previous paper [5]. As the contraposition, a conformal diffeomorphism of M onto M^* has to commute the product structures F and G everywhere in M, and an example of such a conformal diffeomorphism was given in [5].

Outline of the proof. Let M_1 and M_2 be of dimension n_1 and n_2 respectively, $n_1+n_2=n$, and (x^i, x^p) a separate coordinate system of M, (x^i) belonging to M_1 and (x^p) to M_2. Latin indices run on

$$i, j, k=1, 2, \cdots, n_1;\ p, q, r=n_1+1, \cdots, n$$

respectively, and Greek indices κ, λ, μ, ν on the range 1 to n. The metric tensor $g=(g_{\mu\lambda})$ of M has pure components g_{ji} and g_{qp} only with respect to the separate coordinate system (x^i, x^p).

A conformal diffeomorphism f of M to M^* is characterized by a change of the metric tensors

$$g^*_{\mu\lambda}=\frac{1}{\rho^2}g_{\mu\lambda}, \tag{1}$$

ρ being a positive-valued scalar field. The integrability of the product structure G with respect to g^* in M^* is equivalent to

$$\nabla_\mu G_{\lambda\kappa}=-\frac{1}{\rho}(G_{\mu\lambda}\rho_\kappa+G_{\mu\kappa}\rho_\lambda-g_{\mu\lambda}G_{\kappa\nu}\rho^\nu-g_{\mu\kappa}G_{\lambda\nu}\rho^\nu), \tag{2}$$

where ∇ indicates covariant differentiation in M and $\rho_\lambda=\nabla_\lambda\rho$, $\rho^\nu=\rho_\lambda g^{\lambda\nu}$. Denote the gradient vector field (ρ^λ) by Y, the parts (ρ^i) along to M_1 by Y_1 and (ρ^p) to M_2 by Y_2. Put $\Phi=|Y|^2=\rho_\lambda\rho^\lambda=|Y_1|^2+|Y_2|^2$ and