88. On the Galois Cohomology Groups of C_K/D_K

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1. Let k be an algebraic number field and K be its finite Galois extension of degree n with the group G. We denote by C_K and D_K the idele class group of K and its connected component of the unity respectively. In this note, we shall determine the structure of the cohomology group $H^p(G, C_K/D_K)$ for non-negative integer p. For cohomology groups and the morphisms concerned with them, we shall use the notation and terminology as is given in S. Iyanaga [3].

2. In this section, p denotes an arbitrary integer. Let us denote the idele group of K by J_K and its connected component of the unity by H_K. We denote by E the set of all imaginary places of K. Then the maximal compact subgroup of H_K is given by $H'_K=\{x=(x_\mathfrak{p})\in J_K \mid x_\mathfrak{p}=1 \text{ if } \mathfrak{p}\notin E, |x_\mathfrak{p}|=1 \text{ if } \mathfrak{p}\in E\}$. Let us denote the canonical homomorphism from J_K to C_K by φ and $\varphi(H'_K)$ by D'_K. Then we have the following exact sequence

$$(1)\qquad 1 \longrightarrow H'_K \xrightarrow{\varphi} C_K \xrightarrow{\psi} C_K/D'_K \longrightarrow 1.$$

By cohomology sequences belonging to (1) and the fact that $H^q(G, H'_K)=0$ if q is odd, we have

$$(2)\qquad \begin{array}{l} 0 \longrightarrow H^{2p+1}(C_K) \longrightarrow H^{2p+1}(C_K/D'_K) \longrightarrow H^{2p+2}(H'_K) \\ \longrightarrow H^{2p+2}(C_K) \longrightarrow H^{2p+2}(C_K/D'_K) \longrightarrow 0 \qquad \text{(exact).} \end{array}$$

Here we have abbreviated $H^q(G, A)$ to $H^q(A)$ for a G-module A.

Since D_K/D'_K is uniquely divisible, we obtain the isomorphism

$$(3)\qquad H^p(G, C_K/D_K) \cong H^p(G, C_K/D'_K).$$

Hereafter, by virtue of (3), we shall only be concerned with the determination of $H^p(G, C_K/D'_K)$ instead of $H^p(G, C_K/D_K)$.

Let $\{\mathfrak{p}_i \mid 1\leq i\leq r\}$ be the set of all real places of k which ramify in K. If $r=0$, it follows from (2) that

$$H^p(G, C_K/D'_K) \cong H^p(G, C_K) \cong H^{p-2}(G, Z).$$

Therefore, in the following, we exclude this case and shall treat only the case $r>0$. This implies that n is even, so we put $m=n/2\in Z$.

Let $\mathfrak{P}_i$ be one of the extensions of $\mathfrak{p}_i$ to K, and N_i be the decomposition group of $\mathfrak{P}_i$. Let us denote the transfer homomorphism from N_i to G and the restriction from G to N_i on cohomology groups by $\tau^{N_i,G}$ and ρ^{G,N_i} respectively. Since $H^{2p}(G, H'_K)$ is generated by $\tau^{N_i,G}H^{2p}(N_i, H'_K)$, we obtain the following lemma.