85. *Class Number Calculation and Elliptic Unit. III*

Sextic Case

By Ken NAKAMULA
Department of Mathematics, Tokyo Metropolitan University

(Communicated by Shokichi IYANAGA, M. J. A., Sept. 12, 1981)

In our preceding notes [2] and [3], we have introduced an effective method to calculate the class number of a certain cubic or quartic field utilizing its elliptic unit. In the following, we shall treat the same problem for a sextic field.

Let K be a real sextic number field which is not totally real and which contains a (real) quadratic subfield K_2 and a cubic subfield K_3. Let $D\,(>0)$, h and E_+ respectively be the discriminant, the class number and the group of positive units of K. Further, let h_2 and h_3 be the class numbers of K_2 and K_3 respectively. We shall give a way to compute h/h_2h_3 and E_+ at a time by using the "elliptic unit" of K.

§ 1. Illustration of algorithm. Let η_2 and η_3 be the fundamental units (>1) of K_2 and K_3 respectively, and let H_+ be the group of positive units of K, i.e.

$$H_+:=\{\varepsilon\in E_+\,|\,N_{K/K_2}(\varepsilon)=N_{K/K_3}(\varepsilon)=1\}.$$

Then, as in [1], there is the relative fundamental unit ε_1 (>1) in H_+, i.e. $H_+=\langle\varepsilon_1\rangle$, and ε_1 generates E_+ together with two other independent units. More precisely,

$$E_+=\langle\varepsilon_1\rangle\times\langle\varepsilon_2\rangle\times\langle\varepsilon_3\rangle$$

with

(1) $$\varepsilon_2=\sqrt[3]{\eta_2},\quad \sqrt[3]{\eta_2^{\pm1}\varepsilon_1}\quad \text{or } \eta_2,$$

(2) $$\varepsilon_3=\sqrt{\eta_3\varepsilon_1}\quad \text{or } \eta_3.$$

Let η be the elliptic unit of K, of which the definition will be given in § 5. Then, applying the results in Schertz [5], we see that $\eta>1$ and $\eta\in H_+$, and obtain the following formula:

(3) $$h/h_2h_3=(E_+:\langle\varepsilon_1,\eta_2,\eta_3\rangle)(H_+:\langle\eta\rangle)/6.$$

Therefore, the calculation of h/h_2h_3 is reduced to the determination of the group index $(H_+:\langle\eta\rangle)$ and that of the units ε_2, ε_3. The index $(H_+:\langle\eta\rangle)$ is determined similarly as in [2] or [3] by using Theorems 1 and 2 below. The computation of ε_2 and ε_3 is explained in § 4.

§ 2. Upper bound of h/h_2h_3. The following lemma gives an upper bound of the index of a subgroup of H_+.

Lemma 1. *Let $1<\varepsilon\in H_+$ and $D(\varepsilon)$ be the discriminant of ε. Then*