82. "Borel" Lines for Meromorphic Solutions of the Difference Equation $y(x+1)=y(x)+1+\lambda/y(x)$

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1. Introduction. In connection with the iteration of analytic functions, Kimura [1], [2] considered the equation

(E) $$y(x+1)=y(x)+1+\lambda/y(x), \qquad \lambda\neq 0,$$

and obtained a meromorphic solution $\phi(x)$ such that

(1.1) $$\begin{cases} \phi(x)\sim x\Big[1+\sum_{j+k\geqq 1} p_{jk}x^{-j}\Big(\dfrac{\log x}{x}\Big)^k\Big] \qquad (p_{01}=\lambda) \\ \text{in the domain } D_l(R,\varepsilon)=\Big\{|x|>R,\ |\arg x-\pi|<\dfrac{\pi}{2}-\varepsilon\Big\}\cup\{\mathrm{Im}\,[xe^{-i\varepsilon}] \\ >R\}\cup\{\mathrm{Im}\,[xe^{i\varepsilon}]<-R\}, \text{ where } p_{10}=c \text{ is an arbitrarily pre-} \\ \text{scribed constant, } \varepsilon>0, \text{ and } R \text{ is a sufficiently large number} \\ \text{depending on } c \text{ and } \varepsilon. \end{cases}$$

We studied some properties of the solution $\phi(x)$ in [3] and, especially, proved that there is a horizontal line $L=\{\mathrm{Im}\,x=\eta\}$ such that, for any $\delta>0$, in the half strip

(1.2) $$\{x\,;\,|\mathrm{Im}\,x-\eta|<\delta,\ \mathrm{Re}\,x>0\},$$

$\phi(x)$ takes every value infinitely often if $\lambda\neq 1$, and $\phi(x)$ takes every value other than -1 if $\lambda=1$.

We will call such a line as a *"Borel" line* for $\phi(x)$ [4]. It would be natural to inquire how many "Borel" lines may appear for $\phi(x)$.

Our aim in this note is to answer (partially) to this question. We will prove the following

Theorem. *Suppose λ is real in the equation* (E).

(i) *If $\lambda\leqq 1/4$, then there is only one "Borel" line for $\phi(x)$.*

(ii) *If $\lambda>1/4$, then there are at least two "Borel" lines for $\phi(x)$.*

2. Proof of Theorem (i). Let x_0 be a zero point of $\phi(x)$: $\phi(x_0)=0$. Write $x_n=x_0-n$, $n=0,1,\cdots$. Then $\phi(x_1)$ satisfies $0=\phi(x_1)+1+\lambda/\phi(x_1)$, i.e.,

(2.1) $$\phi(x_1)=\frac{1}{2}[-1\pm\sqrt{1-4\lambda}].$$

More generally

(2.2) $$\phi(x_n)=\frac{1}{2}[-(1-\phi(x_{n-1})\pm\sqrt{(1-\phi(x_{n-1}))^2-4\lambda}], \qquad n=1,2,\cdots.$$