76. *Some Explicit Formulae in the Theory of Numbers*

A Remark on the Riemann Hypothesis

By Akio FUJII

Department of Mathematics, Rikkyo University

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§1. **Introduction.** We put for $x \geqq 1$ and for $0<\alpha \leqq 1$,

$$H(x,\alpha)=\sum_{1\leqq n\leqq x}\sum_{1\leqq h\leqq \alpha n}\log(h,n)-\alpha\sum_{1\leqq n\leqq x}\sum_{1\leqq h\leqq n}\log(h,n)+\frac{1}{2}\sum_{n\leqq x}\log n,$$

where (h,n) is the greatest common divisor of h and n. We put for irrational α

$$F(x,\alpha)=H(x,\alpha)+xZ_{\alpha}(1)$$

and for rational $\alpha=a/q$ with $(a,q)=1$,

$$F(x,\alpha)=H(x,a/q)-(x/2q)\log(x/q)+x(\lambda_1(a/q)+\lambda_2(a/q)) -(x/q)(\lambda_3(a/q)+\lambda_4(a/q)),$$

where we put

$$\lambda_j(a/q)=\sum_{b=1}^{q}\left(\left\{\frac{ab}{q}\right\}-\frac{1}{2}\right)\nu_j(b) \qquad \text{for } 1\leqq j\leqq 4,\ \nu_1(b)=1/b,$$

$\nu_2(b)=1/(b+q)$, $\nu_3(b)=\log(1+b/q)$ and $\nu_4(b)=2+\gamma_0-\gamma_{b/q}$ with

$$\gamma_\eta=-\int_1^\infty\frac{\{y\}dy}{(y+\eta)^2},$$

$\{y\}$ is the fractional part of y and $Z_{\alpha}(1)$ is defined below. Under these notations we have shown in [2] the following two theorems which are stated in a slightly different way.

Theorem 1. *The Riemann Hypothesis is equivalent to the statement that for any positive* ε *and for* $X>X_0$,

$$\int_0^1|F(X,\alpha)|^2\,d\alpha \ll X^{1+\varepsilon}.$$

Theorem 2. *Let* Q *be an integer* $\geqq 1$. *Let* $f_1, f_2, \cdots, f_i, \cdots, f_A$ *be the Farey series of order* Q, *namely,* $f_i=a_i/q_i$ *with integral* a_i *and* q_i, $(a_i,q_i)=1$, $0<a_i\leqq q_i$, $0<q_i\leqq Q$ *and* $f_1<f_2<\cdots<f_A$. *Then the Riemann Hypothesis is equivalent to the statement that for any positive* ε *and for* $Q>Q_0$,

$$\sum_{i=1}^{A}|F(Q,a_i/q_i)|^2\ll Q^{3+\varepsilon}.$$

In fact, the gap between above Theorem 1 and our previous Theorem 1 in [2] can be filled by the proof of Lemma 3 below and the gap between above Theorem 2 and our previous Theorem 2 in [2] will be filled in §2. The purpose of the present note is to give, by the classical methods, an explicit relation between $F(X,\alpha)$ for an individual