48. *The Application of Monodromy Preserving Deformation to the Gravitational Field Equation*

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§0. In this note, we will show the new method for constructing exact solutions of the vacuum Einstein equation for stationary axisymmetric gravitational fields (VESA).

From a viewpoint of the inverse scattering theory, Belinsky-Zakharov (B-Z) [1], [2] gave an interesting method for integrating VESA, expressed by the metric form

(0.1) $$-ds^2=f(d\rho^2+dz^2)+g_{\alpha\beta}dx^\alpha dx^\beta \qquad (\alpha, \beta=0, 1)$$

where f and $g_{\alpha\beta}$ are functions in ρ and z, and x^0, x^1 represent the coordinates t, ϕ, respectively.

Under the supplementary condition

(0.2) $$\det g=-\rho^2, \qquad g=(g_{\alpha\beta}),$$

the fields equation for the metric (0.1) can be written as follows:

(0.3) $$\begin{cases} U_\rho+V_z=0 \\ U_z-V_\rho+\rho^{-1}V+\rho^{-1}[U, V]=0 \end{cases}$$

(0.4) $$\begin{cases} (\log f)_\rho=-\rho^{-1}+(4\rho)^{-1}\,\mathrm{trace}\,(U^2-V^2) \\ (\log f)_z=(2\rho)^{-1}\,\mathrm{trace}\,(UV). \end{cases}$$

Here $U=\rho g_\rho g^{-1}$, and $V=\rho g_z g^{-1}$. We should note that the matrix g is symmetric. B-Z found that the equation (0.3) are equivalent to the compatibility conditions of the system of linear equations

(0.5) $$\begin{cases} D_1 Y=\dfrac{\rho V-\lambda U}{\lambda^2+\rho^2}Y, \\ D_2 Y=\dfrac{\lambda V+\rho U}{\lambda^2+\rho^2}Y, \end{cases}$$

where

$$D_1=\frac{\partial}{\partial z}-\frac{2\lambda^2}{\lambda^2+\rho^2}\frac{\partial}{\partial \lambda}, \qquad D_2=\frac{\partial}{\partial \rho}+\frac{2\lambda\rho}{\lambda^2+\rho^2}\frac{\partial}{\partial \lambda},$$

and λ is a complex parameter independent of ρ and z.

If we find a solution $Y(\lambda)=Y(\lambda, \rho, z)$ to (0.4), and set

(0.6) $$g=Y(0)=Y(0, \rho, z),$$

the potentials U and V in (0.5) can be recovered as $U=\rho g_\rho g^{-1}$, $V=\rho g_z g^{-1}$, so we obtain a solution of (0.3). But we should note that the function g given by (0.6) is not always assured to be symmetric, real, and to satisfy the condition (0.2). We can easily find the conditions that g is real and satisfies (0.2) (cf. [1], [2], [9]). Therefore one of the