38. *On the Asymptotic Behavior of Asymptotically Nonexpansive Semi-Groups in Banach Spaces*

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1. Introduction and statement of results. Throughout this paper X denotes a *uniformly convex* real Banach space and C is a nonempty *closed* subset of X. Let J be an unbounded subset of $[0, \infty)$ such that

(1.1) $$t+s \in J \quad \text{for every } t, s \in J,$$

and

(1.2) $$t-s \in J \quad \text{for every } t, s \in J \text{ with } t>s.$$

A family $\{T(t): t \in J\}$ of mappings from C into itself is called an *asymptotically nonexpansive semi-group* on C if

(1.3) $$T(t+s)=T(t)T(s) \quad \text{for } t, s \in J$$

and there exists a function $a: J \to [0, \infty)$ with $\lim_{t\to\infty} a(t)=1$ such that

(1.4) $$\|T(t)x-T(t)y\| \le a(t)\|x-y\| \quad \text{for every } x, y \in C \text{ and } t \in J.$$

In particular if $a(t)\equiv 1$, then $\{T(t): t \in J\}$ is called a *nonexpansive semi-group* on C. The set of fixed points of $\{T(t): t \in J\}$ will be denoted by F, i.e. $F=\{x \in C: T(t)x=x \text{ for all } t \in J\}$. We denote by $C_{11}[0, \infty)$ ($C_1[0, \infty)$) the set of increasing (nondecreasing) continuous functions defined on $[0, \infty)$.

In this paper we deal with the strong convergence of trajectories of semi-groups. Our first result is the following which extends and unifies several results in [1], [2], [4].

Theorem 1. *Let $\{T(t): t \in J\}$ be an asymptotically nonexpansive semi-group on C with $F \ne \phi$, and let $x \in C$. Suppose that*

(a$_1$) *there exist $x_0 \in F$, $\varphi \in C_{11}[0, \infty)$, $\psi \in C[0, \infty)$ and a nonnegative function b defined on J with $\lim_{h\to\infty} b(h)=1$ such that*

$$\varphi(\|T(h)u+T(h)v-2x_0\|) \le \varphi(b(h)\|u+v-2x_0\|)+[\psi(b(h)\|u-x_0\|) - \psi(\|T(h)u-x_0\|)+\psi(b(h)\|v-x_0\|)-\psi(\|T(h)v-x_0\|)]$$

for every $u, v \in \{T(t)x: t \in J\}$ and $h \in J$ and

(a$_2$) $$\lim_{t\to\infty} \|T(t+h)x-T(t)x\|=0 \quad \textit{for every } h \in J.$$

Then $\{T(t)x: t \in J\}$ converges strongly as $t \to \infty$ to an element of F.

Remark. Suppose that $T: C \to C$ is nonexpansive (i.e. $\|Tu-Tv\|$

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