101. *On a Difference System of the Integrals of Pochhammer*

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In this note we investigate a difference system of the integrals of Pochhammer

$$P_c(\hat{\lambda})=\int_c(\zeta-a_1)^{\lambda_1}\cdots(\zeta-a_n)^{\lambda_n}d\zeta, \tag{1}$$

with respect to the variable $\hat{\lambda}=(\lambda_1,\cdots,\lambda_n)$, for a suitable cycle C. As is well-known, all the functions $P_c(\hat{\lambda}+\hat{k})$ $(\hat{k}\in Z^n)$ are expressed as linear combinations, with rational coefficients of $\hat{\lambda}$, in terms of $u_k(\hat{\lambda})=P_c(\hat{\lambda}-\hat{e}_k)$ $k=1,\cdots,n$, where $\hat{e}_k$ is the unit vector $(0,\cdots,\overset{k-th}{1},\cdots,0)$ (cf. [3, § 18.26]). The difference system is determined by the following

$$u_i(\hat{\lambda}-\hat{e}_j)=(a_i-a_j)^{-1}(u_i(\hat{\lambda})-u_j(\hat{\lambda}))\qquad i\neq j, \tag{2}$$

with the fundamental relation

$$\sum_{i=1}^n \lambda_i u_i(\hat{\lambda})=0. \tag{3}$$

The system (2) and (3) defines an element of a cocycle belonging to the cohomology $H^1(Z^n, GL_{n-1}(\mathbf{C}(\hat{\lambda})))$. But the structure of $H^1(Z^n, GL_{n-1}(\mathbf{C}(\hat{\lambda})))$ for $n\geqq 3$ seems generally very difficult to determine. Therefore, we consider the system of the following special type

$$u_i(\hat{\lambda}-\hat{e}_k)=\sum_{j=1}^n b_{ij}^k u_j(\hat{\lambda})\qquad i\neq k, \tag{4}$$

with the fundamental relation

$$\sum_{i,j=1}^n a_{ij}\lambda_i u_j(\hat{\lambda})=0, \tag{5}$$

where a_{ij} and b_{ij}^k $(k=1,\cdots,n)$ denote constant matrices of rank n and $n-1$, respectively.

Theorem 1 (A characterization of the Pochhammer system). *Suppose that the system* (4) *and* (5) *has* $(n-1)$ *linearly independent solutions which are meromorphic with respect to* $\hat{\lambda}$. *Then this system becomes* (2) *and* (3), *except for a constant multiple of each* $u_i(\hat{\lambda})$ $(i=1,\cdots,n)$.

From now on, we shall assume that $a_1,\cdots,a_n$ are real numbers such that $a_1<\cdots<a_n$. In the case of several variables, when we restrict ourselves to asymptotic expansions only in "rational directions", the solution of (2) is completely determined by a difference system of one variable ([1, Théorème 1.2]).