99. *On a Stability of Essential Spectra of Laplace Operators on Non-Compact Riemannian Manifolds*

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§1. **Introduction.** Let M be an n-dimensional Riemannian manifold, g its Riemannian metric and Δ_g the Laplace operator associated to g. If M is compact, it is well known that Δ_g is essentially selfadjoint in $L_2(M, \mathrm{d}_g x)$, where $\mathrm{d}_g x$ is the volume element associated to g. Also the spectrum $\sigma(\Delta_g)$ of Δ_g consists of only isolated eigenvalues with finite multiplicities. On the other hand, if M is not compact, Δ_g has in general many selfadjoint extensions, and the spectrum may contain continuous part or eigenvalues with infinite multiplicities. In the first case, under a deformation of a Riemannian metric, the eigenvalues move continuously in a certain sense. In this note we concern ourselves with essential spectrum of Δ_g for a non-compact manifold. We show the following

Theorem. *Let (M, g) be a Riemannian manifold. Assume that Δ_g is essentially selfadjoint. Let g_1 be another Riemannian metric which is different from g only on a compact subset K of M. Then,*

(i) *Δ_{g_1} is also essentially selfadjoint in $L_2(M, \mathrm{d}_{g_1} x)$,*

(ii) *the essential spectrum of Δ_g is contained in the spectrum $\sigma(\Delta_{g_1})$ of Δ_{g_1}.*

Here the essential selfadjointness of Δ_g means that the closure $\bar{\Delta}_g$ in $L_2(M, \mathrm{d}_g x)$ of Δ_g acting on $C_0^\infty(M)$ is selfadjoint. In this case, it is easy to show that it coincides with the extension of Δ_g in the sense of distribution, that is, the domain D of $\bar{\Delta}_g$ consists of those $\phi \in L_2(M, \mathrm{d}_g x)$ such that $\Delta_g \phi \in L_2(M, \mathrm{d}_g x)$.

For selfadjoint operators the spectrum can be divided into two parts, the one consisting of all isolated eigenvalues with finite multiplicities and the other, remaining set, called the essential spectrum. The following proposition is known (see [3, p. 518]).

Proposition. *Let λ be in the essential spectrum of a selfadjoint operator A on a Hilbert space H. Then there exists an orthonormal sequence $\{x_n\}_{n\geq 1}$ in H such that*

$$\|(A-\lambda)x_n\| \to 0, \qquad \text{as } n\to\infty.$$

§2. **Proof of the theorem.** For $\phi \in L_2(M, \mathrm{d}_g x)$ we denote its norm by $\|\phi\|$. Let U be an open subset of M such that the closure $\bar{U}$ is