86. *Polynomial Hamiltonians associated with Painlevé Equations. II*[*)]

Differential equations satisfied by polynomial Hamiltonians

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1. Introduction. The present article concerns the polynomial Hamiltonians associated with the six Painlevé equations. The notation of the previous note [1] will be adopted throughout this paper; we will refer to the Painlevé equation as P_J ($J=\mathrm{I}, \cdots, \mathrm{VI}$) and denote by H_J the polynomial Hamiltonian $H_J(t;\lambda,\mu)$ associated with P_J, given in Table (**H**) of [1]. Let Ξ_J be the set of fixed critical points of P_J and let $\tilde{\boldsymbol{B}}_J$ be the universal covering surface of $\boldsymbol{B}_J=\boldsymbol{P}^1(\boldsymbol{C})-\Xi_J$. Any solution $(\lambda(t),\mu(t))$ of the Hamiltonian system with the Hamiltonian $H=H_J$,

$$(1)\qquad \begin{cases} \lambda'=\dfrac{\partial H}{\partial \mu} \\ \mu'=-\dfrac{\partial H}{\partial \lambda}, \end{cases}$$

is meromorphic on $\tilde{\boldsymbol{B}}_J$ and so is the function defined by

$$(2)\qquad H_J(t)=H_J(t;\lambda(t),\mu(t)).$$

The τ-function $\tau=\tau_J(t)$ related to $H_J(t)$ is defined by

$$(3)\qquad H_J(t)=\frac{d}{dt}\log\tau_J(t),$$

and it is holomorphic on $\tilde{\boldsymbol{B}}_J$ ([1]).

2. Equation $P_{\mathrm{III}'}$. Consider firstly the equation

$$P_{\mathrm{III}'}\qquad \lambda''=\frac{1}{\lambda}(\lambda')^2-\frac{1}{t}\lambda'+\frac{\lambda^2}{4t^2}(\gamma\lambda+\alpha)+\frac{\beta}{4t}+\frac{\delta}{4\lambda}.$$

We assume that none of γ and δ is zero. In [2], Painlevé showed that $P_{\mathrm{III}'}$ is the limiting form of the equation P_{V} and is transformed to P_{III} by the change of variables: $t\to t^2$, $\lambda\to t\lambda$. Furthermore, we can derive from H_{V} the polynomial Hamiltonian associated with $P_{\mathrm{III}'}$,

$$H_{\mathrm{III}'}\qquad \frac{1}{t}\left[\lambda^2\mu^2-(\eta_\infty\lambda^2+\theta_0\lambda-\eta_0 t)\mu+\frac{1}{2}\eta_\infty(\theta_0+\theta_\infty)\lambda\right],$$

by a process of coalescence. Here the constants in $H_{\mathrm{III}'}$ are related to $\alpha,\beta,\gamma,\delta$ as follows:

$$\alpha=-4\eta_\infty\theta_\infty,\quad \beta=4\eta_0(\theta_0+1),\quad \gamma=4\eta_\infty^2,\quad \delta=-4\eta_0^2.$$

It follows from the assumption $\gamma\delta\neq 0$ that none of η_Δ ($\Delta=0,\infty$) is zero.

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