82. *On Certain Densities of Sets of Primes*

By Leo MURATA
Department of Mathematics, Faculty of Science,
Tokyo Metropolitan University

(Communicated by Shokichi IYANAGA, M. J. A., Sept. 12, 1980)

Let $\mathcal{P}$ be the set of all rational primes and M a non-empty subset of $\mathcal{P}$. For a pair of real numbers (α,β), where either $\alpha=0$ and $\beta\geqq 0$ or $\alpha>0$ (β arbitrary), and for positive x, put $f_{\alpha,\beta}(x)=x^{\alpha-1}(\log x)^{\beta}$. We put furthermore

$$\pi_{\alpha,\beta}(M,x)=\sum_{p\in M,\, p\leqq x} f_{\alpha,\beta}(p),$$
$$d_{\alpha,\beta}(M,x)=\frac{\pi_{\alpha,\beta}(M,x)}{\pi_{\alpha,\beta}(\mathcal{P},x)},$$
$$\underline{D}_{\alpha,\beta}(M)=\liminf_{x\to\infty} d_{\alpha,\beta}(M,x),$$
$$\bar{D}_{\alpha,\beta}(M)=\limsup_{x\to\infty} d_{\alpha,\beta}(M,x).$$

When $\underline{D}_{\alpha,\beta}(M)=\bar{D}_{\alpha,\beta}(M)$, we denote this value by $D_{\alpha,\beta}(M)$ and say that M has the (α,β)-*density* $D_{\alpha,\beta}(M)$. The *natural density* is nothing other than $(1,0)$-density and, as is well-known, the *Dirichlet density* is equal to $(0,0)$-density (cf. [1]).

We shall say that (α,β)-density is *stronger* than (γ,δ)-density, and write $D_{\gamma,\delta}\prec D_{\alpha,\beta}$, if the existence of $D_{\alpha,\beta}(M)$ for $M\subset\mathcal{P}$ implies the existence of $D_{\gamma,\delta}(M)$ and, when these densities exist, their values are the same ($\prec$ is obviously an order relation). If $D_{\alpha,\beta}\prec D_{\gamma,\delta}$ and $D_{\gamma,\delta}\prec D_{\alpha,\beta}$, we say that both densities are *equivalent* and write $D_{\alpha,\beta}\sim D_{\gamma,\delta}$ ($\sim$ is clearly an equivalence relation).

Our main theorem states:

Theorem 1. *Any of our (α,β)-densities is equivalent to one of the three densities, $D_{0,0}, D_{0,1}, D_{1,0}$, which will be denoted by d_0, d_1, d_2, respectively. We have furthermore $d_0\prec d_1\prec d_2$ and these three densities are inequivalent.*

As noted above, d_0 and d_2 are Dirichlet density and natural density, respectively. It is known that $d_0\prec d_2$ (cf. [1]). Our theorem shows that the density d_1 lies, so to speak, between the two.

The following theorem gives a more precise form of the first part of Theorem 1.

Theorem 2. *For any $\beta>0$, $D_{0,\beta}$ is equivalent to $d_1=D_{0,1}$ and for any $\alpha>0$ and any β, $D_{\alpha,\beta}$ is equivalent to $d_2=D_{1,0}$.*

Sketch of proof of Theorem 2. It is easily shown that