78. *On the Hessian of the Square of the Distance on a Manifold with a Pole*

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Analysis on a manifold with a pole has been studied in a series of papers by Greene-Wu. In particular, the characterization of C^n in terms of geometric conditions is one of the most interesting problems. In the case of a simply-connected complete Kähler manifold of non-positive curvature this problem has been solved by Siu-Yau [2] and Greene-Wu [1] (Theorem J). Concerning these results Wu has proposed some open problems in [4] and [5]. In this note we shall prove theorems related to his propositions. The author would like to thank Prof. Wu whose suggestion made this note materialize.

1. A smooth mapping $\phi: N \to M$ between Riemannian manifolds is called a *quasi-isometry* iff ϕ is a diffeomorphism and there exist positive constants μ and ν such that for each tangent vector X on N,

$$\mu|X|_N \leqq |\phi_*(X)|_M \leqq \nu|X|_N.$$

We recall that (M, o) is called a *manifold with a pole* iff M is a Riemannian manifold and the exponential mapping at $o \in M$ is a global diffeomorphism. Let (M, o) be a manifold with a pole. The distance function from the pole o will be denoted by r so that r^2 is a smooth function on M. The first theorem in question is the following

Theorem 1. *Let (M, o) be a manifold with a pole. Suppose there exists a continuous non-negative function $\varepsilon(t)$ on $[0, \infty)$ such that:*

(1) $|(1/2)D^2r^2 - g| \leqq \varepsilon(r)g$,

(2) $\varepsilon_o = \int_0^\infty (\varepsilon(t)/t)dt < \infty$.

Then $\exp: T_o(M) \to M$ *is a quasi-isometry satisfying*

$$\exp(\varepsilon_o)^{-1}|V| \leqq |\exp_*(V)| \leqq \exp(\varepsilon_o)|V|$$

for any tangent vector V at any point in $T_o(M)$.

In (1) above, D^2r^2 denotes the Hessian of the smooth function r^2 on M. Moreover inequality (1) means the following: If $x \in M$ and $X \in T_x(M)$ is a unit vector, then

$$\left|\frac{1}{2}D^2r^2(X, X) - 1\right| \leqq \varepsilon(r(x)).$$

Remark. It follows from the above theorem that if (M, o) is a manifold with a pole and $(1/2)D^2r^2 = g$ on M then M is isometric to a Euclidian space. This is a weak form of a theorem by H. W. Wissner