65. On the Linear Sieve. I

By Yoichi MOTOHASHI
Department of Mathematics, College of Science and Technology, Nihon University

(Communicated by Kunihiko KODAIRA, M. J. A., June 12, 1980)

1. Two different proofs of the linear sieve are known: One is due to Jurkat-Richert [6] and the other to Rosser (unpublished, but see [8]) and Iwaniec [2] (see also [3]). Comparing these proofs one may note that Jurkat-Richert's procedure is simpler than that of Iwaniec in the treatment of the convergence problem arising from the infinite iteration of the Buchstab identity. But the Rosser-Iwaniec sieve has the important advantage that it admits a very flexible bilinear form for the error-term; this was discovered by Iwaniec [4] and must be a milestone in the sieve history as its applications (cf. [5]) indicates clearly. It seems unlikely, however, that the similar improvement may be introduced to the Jurkat-Richert sieve; the reason for this lies in their use of the Selberg sieve as an aid.

Now the purpose of this note is to show briefly that one may reduce considerably the aforementioned difficulty in the Rosser-Iwaniec sieve by combining an important idea of Jurkat-Richert [6] with the Rosser truncation of the Buchstab identity.*) The details will be given in [7], and here we indicate only the clues.

Acknowledgement. The present author is very much indebted to his friend Dr. H. Iwaniec for sending him various important unpublished manuscripts.

2. Now let A be a finite sequence of integers and P a set of primes. Let $S(A, z)=|\{a \in A \mid (a, P(z))=1\}|$, where $P(z)=\prod p$ over $p<z$, $p \in P$. Let $A_d=\{a \in A \mid a\equiv 0 \pmod{d}\}$ and put $R_d=|A_d|-X\delta(d)/d$, where X is a parameter and δ a multiplicative function. As in [1] we introduce the condition $\Omega_2(1, L)$: For any $2\le u\le v$

$$-L\le \sum_{\substack{u\le p<v\\ p\in P}} \frac{\delta(p)}{p}\log p-\log\frac{w}{v}\le C,$$

where L is a parameter and C a constant. Next we define functions F and f by $F(u)=2e^{\gamma}/u$, $f(u)=0$ if $0<u\le 2$ and by $(u\,f(u))'=F(u-1)$, $(u\,F(u))'=f(u-1)$ if $u>2$, where γ is the Euler constant; also we put $\phi_\nu(u)=F(u)$ if ν is odd, and $\phi_\nu(u)=f(u)$ if ν is even. Finally we denote by $E(y)$ the sum $\sum|R_d|$ over $\mathrm{d}<y$, $d|P(z)$, y being another parameter.

*) It seems that this confirms partially Selberg's anticipation expressed at the bottom lines of [8, p. 343].